Teaching Derivations of Area and Measurement Concepts of the Circle: A Conceptual-Based Learning Approach through Dissection Motion Operations

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Abstract
In this article a variety of Dissection Motion Operations (DMO) are presented; they are primarily focused at the derivation of the area formula of the circle through hands-on manipulation for the classroom practices in a *conceptually-based learning fashion*. The National Council of Teachers of Mathematics (NCTM, 2000), within its dedication to the improvement of mathematics pedagogy and assessment, has established the standards for mathematics teachers and policymakers in the United States and beyond. Adopted by the Ontario Ministry of Education (2005), the NCTM's principles have infiltrated the Ontario mathematics curriculum. As such, *conceptually-based learning of mathematics versus procedural-based learning* has had an opportunity to flourish in the classrooms of the province of Ontario, Canada, for the past six years. The movement toward reform based mathematics classrooms finds its philosophy grounded in the concerns about procedural based learning that is taught without conceptual understanding. Several conceptual-based approaches to the origin of the formulas such as the area of the circle will be presented in this article.

Background
In reference to the pitfalls of the procedural-based learning methodology in teaching, Freire (2003) has raised a red flag about this methodology and has long written about teachers' responsibility to do more than "fill" students with the contents of lectures. Freire has directly talked about the status of teacher-directed education where "banking" concepts through which students were seen to have the responsibilities in regards to the storing, receiving and filing of deposits. Freire (2003) further argued that in the process of such rote learning procedures, it is the students themselves that are being filed away (p. 55). Freire's position has echoed the National Council of Teachers of Mathematics Standards (NCTM, 2000) calling for conceptual meaningful understanding versus rote learning and memorization. This article will investigate the literature as it pertains to the numeracy level of secondary school graduates in Canada. Research indicates that both students and teachers in North America have difficulty with measurements involving area and that this difficulty can be traced to area attribute problems (Baturo and Nason, 1996; Battista, 2003; Kouba, Brown, Carpenter, Lindquist, Silver and Swafford, 1988). Battista (2003) further discussed the lack of "spatial structuring" which is the underpinning of area and volume concepts and which result in the inability of students to apply what they have learned (p. 132). Baturo and Naon (1996) have made a specific emphasis at the use of manipulatives as a medium for hands-on manipulation in addition to the specific *dynamic* aspect of the area concept stating that:

> In those schools where teachers do provide their students with concrete experiences in developing the notion of area and its measurement, it is often done in a cursory and disconnected fashion or, if done conscientiously, tends to focus almost exclusively on the static perspective of the notion of area to the exclusion of the dynamic of area (Baturo and Nason, 1996, p. 239).

Conceptual understanding of the concepts and processes involved in area measurement includes "knowing that the area of a shape remains the same when it has been changed by 'Cutting and Pasting' to form different shapes" for concrete knowledge (Baturo and Nason, 1996, p. 239). Further, conceptual understanding includes "knowing that congruence and 'cut and paste' transformations conserve the area of a shape; area is a continuous attribute that can be divided into discrete subunits" (Baturo and Nason, 1996, p. 236).
Baturo and Nason (1996) further discussed the importance of students understanding of how pi was derived and the shapes it is associated with along with how the formulae associated with each shape are developed and connected. In this connection, Rahim (2010) has introduced a treatment for teaching the derivation of area formulas for polygonal regions through Dissection-Motion-Operations (DMO) as a systematic refined process for "Cutting and Pasting" acts cited above. Baturo and Nason (1996) further stated that students who combined shapes when attempting solving problems were more stimulated than if they were simply finding the areas of the shapes. Alarmingly, their study revealed that only two of their 13 students understood the relationship between the area of a rectangle and the area of a triangle. As a result, their use of the formula for area held no meaning for them. The students admitted that they did not remember engaging in activities that were meant to enhance their understanding of this relationship (p. 256). Stephan and Clements (2003) recognized that maintaining area is an important concept that it is often not included in measurement instruction. The authors stated that students have difficulty to understand that when a shape is cut up into a finite number of pieces and when the pieces are rearranged into a different shape with no overlapping, its area remains the same. Further, Stephan and Clements (2003) found that research indicated that children use differing strategies to conceptually comprehend the ideas behind the concept of area.

Historically, Ma (1999) stated that the approximation of the area of a circle using the area of a parallelogram has been known since the 17^{th} century (p. 116), (see Smith & Mikami, 1914, p. 131). Ma has indicated how Chinese teachers have made a full lesson by inspiring students to subdivide a circle into sectors and rearrange the pieces into a parallelogram like shape, then, by imagining the number of the sectors increases, a closer resemblance of a parallelogram shape will be reached. And as the approximation process continues the ultimate shape ends up into a parallelogram (and hence into a rectangle by further cutting and moving) implying the presence of the area formula of the circle (p. 116).

Teaching the Derivation of the Area of a Circle: A Dissection-Motion-Operation Approach

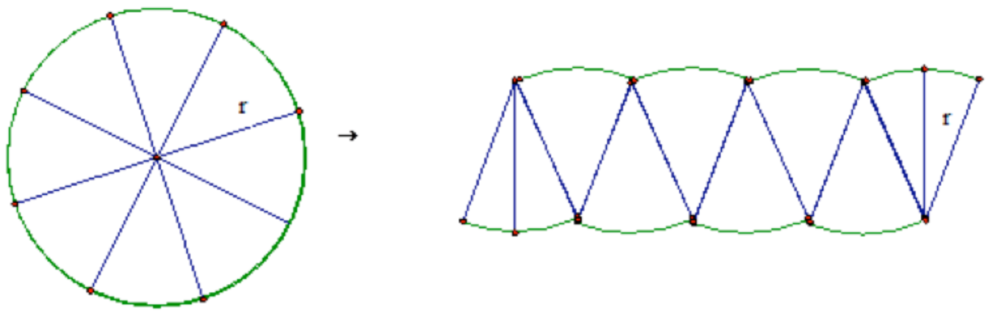
Clearly, the number of equivalent sectors in which a circle can be dissected would be either odd or even. As such, there will be three possibilities of dissecting a circle:
Case 1. The number of the equivalent sectors, n, in which a circle can be divided, is even;
Case 2. The number of the equivalent sectors, n, in which a circle can be divided, is odd; and,
Case 3. The number of equivalent sectors, n (even or odd), is a perfect square.

As Case 1 (will be shown below) deals with the relationship between the circle and the parallelogram, it should be noted that this relationship has been known and used for centuries. Most recent elementary mathematics texts have used this relationship to introduce a visual conceptual understanding of the area of a circle in a dynamic and meaningful fashion.

Incidentally, Rahim (2010) has introduced a decomposition-composition (Dissection-Motion) treatment of showing the origins of all area formulas of polygons of all types. Rahim was focusing at the dynamic teaching approach for the derivation of area formulas for polygonal regions through what is called Dissection-Motion-Operations (p. 195). The following are derivations of the formula for the area of the circle through *shape transforms* for each of the three cases listed below.

Case 1: The number of the equivalent sectors, n, in which a circle can be divided, is ***even***. In this case, the circle is transformed into a ***parallelogram*** like shape of equal area through the following steps:

1. Dissection Step: Consider the dissection of the circle given in Figure 1. Note that, in this case, the circle can be dissected into ***n*** *number of equivalent sectors where n can be any* ***even*** *positive integer* **≥ 4**. However, for simplicity, assume it is dissected into *eight* equivalent sectors as shown in the left part of Figure1 below.
2. Motion Step: Each of the eight individual sectors is appropriately moved under a transformation operation of translation, rotation, and or reflection to form the parallelogram like shape shown at the right part of Figure 1.



|------ ½(Circumference= 2πr) = πr -----|

Figure 1. Shape transform of a circle into a parallelogram resemblance through DMO

Then, from Figure 1, it is clear that

$$\text{Area of the circle} = \text{Area of the parallelogram like shape}$$
$$\approx \tfrac{1}{2}(2\pi r)\times r = \pi r^2.$$

This result holds true when the number of equivalent sectors increases to approach infinity (∞). Accordingly, the shape at the right side of Figure 1 will take the shape of the parallelogram.

Case 2: The number of the equivalent sectors, n, in which a circle can be divided, is *odd*. In this case, the circle is transformed into a ***trapezoid*** like shape of equal area through the following steps:

1. Dissection Step: Consider the dissection of the circle given in Figure 2. Note that, in this case, the circle can be dissected into ***n*** *number of equivalent sectors where n can be any* ***odd*** *positive integer* **≥ 3**. For simplicity, assume it is dissected into five equivalent sectors as shown in the left part of Figure2 below.

2. **Motion Step:** Each of the five individual sectors is appropriately moved under a transformation operation of translation, rotation, and or reflection to form the trapezoid like shape shown at the right part of Figure 2.

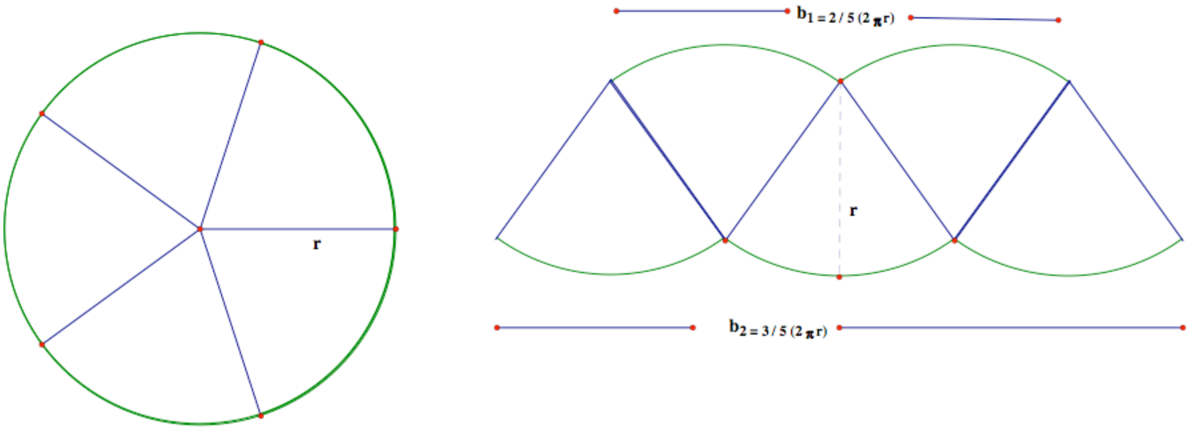


Figure 2. Shape transform of a circle into a trapezoid resemblance through DMO

Then, from Figure 2, it follows that

Area of the circle = Area of the trapezoid like shape

$$\approx \tfrac{1}{2}\,(h)(b_1 + b_2)$$

$$= \tfrac{1}{2}(r)[(2/5)(\mathbf{2\pi r}) + (3/5)(\ \mathbf{2\pi r}) = \mathbf{\pi r^2}.$$

This result holds true when the number of equivalent sectors increases approaching infinity (∞). As a result, the shape at the right of Figure 2 should take the shape of the isosceles trapezoid.

Case 3. The number of equivalent sectors, n (even or odd), is a perfect square. In this case, the circle is transformed into an isosceles triangle like shape of equal area through the following:

1. Dissection Step: Consider the dissection of the circle given in Figure 3. Note that, in this case, the circle can be dissected into ***n** number of equivalent sectors where n can be any* ***perfect square** positive integer* **≥ 4**. Assume the circle is dissected into nine equivalent sectors as shown in the left part of Figure3 below.
2. Motion Step: Each of the nine individual sectors is appropriately moved under a transformation operation of translation, rotation, and or reflection to form the isosceles triangle like shape shown at the right part of Figure 3.

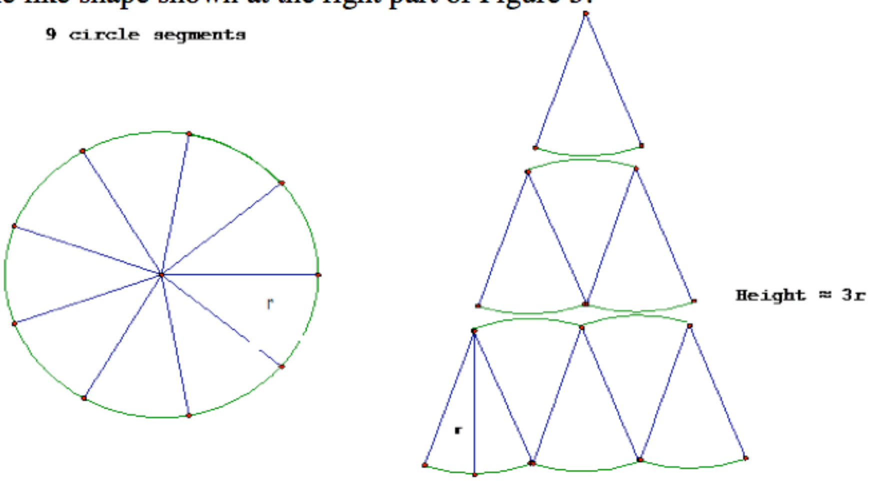


Figure 3. Shape transform of a circle into an isosceles triangle resemblance through DMO

From Figure 3, it follows that

Area of the circle = Area of the isosceles like shape

$\approx \frac{1}{2}$ (h)(b)

$\approx \frac{1}{2}(3r)[(3/9)(\mathbf{2\pi r}) = \mathbf{\pi r^2}.$

And similarly, this result holds true when the number of equivalent sectors increases approaching infinity (∞). Thus the shape at the right side of Figure 3 will take the shape of an isosceles triangle.

Epilogue

Further relationships among 2D shapes can be developed through classroom sessions for building conceptual mathematical understanding of the area and measurement concepts of a variety of shapes in geometry. This type of shapes-to-shape interrelationships is directly related to van Hieles (1985) levels of thought development in geometry. As such, these interrelationships are vital and necessary particularly for middle school levels for a deep understanding of measurement in middle school levels such as grades 8 and 9.

The literature suggests that Canada (and the rest of the world for that matter) is currently struggling with numeracy in general, including in particular, the conceptual understanding of the area of a circle. Battista (2003) has expressed concerns on the lack of spatial comprehension that is resulting in an inability for students to apply what they have learned (p. 131). Ma (1999) and Linquist (1987) have stressed the importance for students to explore multiple perspectives. Stigler and Heibert (1999) have reported how the Chinese teachers in their study were considering frustration and struggle as part of the learning process. On the other hand, Baturo and Nason (1996) reported a failure on the part of teachers in North America to provide concrete experiences that involve the development of the area concept in the classroom. The authors stated further that teachers are teaching "to the exclusion of the dynamic of area" (p. 239). Rahim (1986) have introduced examples of how area can be discussed in a dynamic sense in the classroom.

We have shown in this paper that there are a number of ways for students to discover the area of a circle, offering many opportunities for deep conceptual understanding with which students can move forward along the van Hiele's level of geometric thought development.

The dynamic approach is in no way limited to hands-on manipulations, rather, it can be introduced through the use of Dynamic software too such as the Geometer's Sketchpad (GSP Version 5 – North America-Based software) and Cabri II and Cabri 3D (Grenoble, France-Based software). Furthermore, the dynamic feature in teaching geometric shapes is not restricted to two dimensional objects; it is suitable to 3D objects too. We believe that with a dynamic approach, students are given the opportunity to see themselves as emerging critical thinkers in the process!

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