Tangram-base Problem Solving in Radical Constructivist Paradigm: High School Student-Teachers Conjectures

Medhat H. Rahim, mhrahim@lakeheadu.ca, Radcliffe Siddo, rsiddo@lakeheadu.ca, Moushira Issa, missa@lakeheadu.ca
955 Oliver Road, Lakehead University, Faculty of Education, Thunder Bay, Ontario, Canada P7B 5E1

Abstract
A series of tangram-based problem solving tasks, focusing at visual geometric construction and justification by high school teacher candidates, are reported. Sociocultural and psychological components of von Glasersfeld Theory of *radical constructivism* have been utilized. The purpose was to describe and analyze how students' cognitive constructions have been initiated, modified, and re-modified as they were proceeding in their attempts to solve and justify spatial tangrams-based problems.

Background
The 'tangram' has been originally referred to as the 7-pieces dissection (or tangram problem) consisting of seven flat shapes forming together a square shape (five triangles: two identical large, two identical small and one medium triangles; a small square and a parallelogram). It was originally invented in China at some unknown year in history, and then carried over to the world by trading ships in the early 19th century to become well-known since then (Wang & Hsiung, 1942; Read, 1965). In particular, assuming that the small square has an area of one unit square then each large triangle has an area of two unit square and each small triangle of an area of half unit square and medium triangle of an area of one unit square and a parallelogram have an area of one unit square too. The objective of the tangram problem, often called tangram puzzle, is to form a specific shape, given only its outline or description, using all seven pieces with no overlapping (Figure 1).

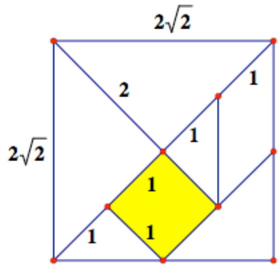


Figure 1: Seven pieces tangram set resembled into a square of side $2\sqrt{2}$ units

Clearly a given tangram set is not confined to a particular rigid and fixed shape formation such as the case of a "jigsaw" puzzle set of pieces, rather the pieces can be rearranged into several geometric shapes each of which is area equivalent to the original tangram square. Meanwhile the new constructed shapes may be convex: a shape is convex if the line joining any two points within the shape falls entirely in the shape; otherwise it is a non-convex shape. This tangram shapes formation represents a rich environment through which shape-to-shape, shape-to- parts and part-to-part interrelationships would be explored by the learners and possibly at all levels of schooling, from pre-kindergarten to university teacher training classes.
The literature provides a variety of examples using the seven pieces tangram set. The pieces may be rearranged into other distinct shapes each of which contains also seven pieces yet are of equal

area of the original seven pieces tangram set whether a resultant shape is convex or not. Below are the *only* thirteen convex shapes that are the resultant of the shape formation using the seven piece tangram (see Wang & Hsiung, 1942, p. 596; Scott, 2006, p. 5).

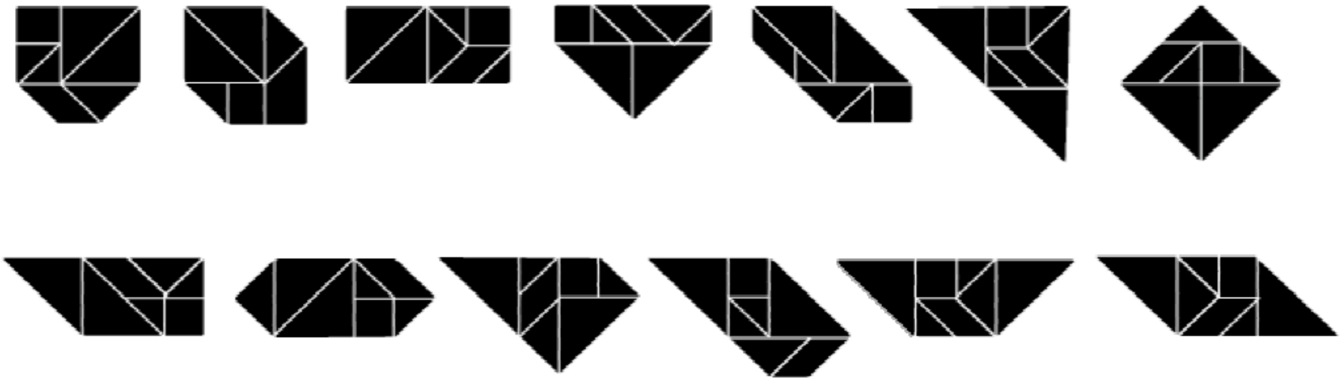


Figure 2: The thirteen convex shapes made out of the seven pieces tangram set

Also, tangram shapes formation may be carried out by using only two, three, four, five or six pieces in addition to the whole seven pieces.

Radical Constructivism: Sociocultural and Psychological Mechanisms in Justification

It has been suggested by von Glasersfeld (1995) that, based on his radical constructivism theory known as "*Radical Constructivism: A Way of Knowing & Learning*", when individuals deal with the physical world, their minds construct, through certain mental mechanisms and collections of cognitive structures, their conceptualization, reason, and coordination of their engagements (von Glasersfeld, 1995; 1984; 1974). Battista (1999) has described the notion of *abstraction* as the process through which the mind selects, coordinates, unifies, and registers in memory a collection of mental acts that appear in the attentional field (p. 418). Further, Battista (1999) has referred to von Glasersfeld's (1995, p. 69) ideas of *abstraction* and added that abstraction has several levels: At its *perceptual level* (most basic), abstraction isolates an item in the stream of an experience and seizes it as a unit. Battista added that material or entity is said to have reached the *internalized level* whenever it has been sufficiently abstracted so that it can be re-presented (re-created) in the absence of its perceptual input. Material or entity is said to have reached *interiorized level* whenever it has been disembodied from its original perceptual context and it can be freely operated on in imagination, including being "projected" into other perceptual material and utilized in novel situations (Battista, 1999, p. 418). Earlier, Steffe & Cobb (1988) asserted that *interiorization* is "the most general form of abstraction; it leads to the isolation of structure (form), pattern (coordination), and operations (actions) from experiential things and activities" (p. 337). von Glasersfeld, in presenting his radical constructivism theory, stated that *understanding* requires more than abstraction; it requires *reflection* which is the conscious process of re-presenting experiences, actions, or mental processes and considering their results or how they are composed. *Reflective abstraction* takes mental operations performed on previously abstracted items as elements and coordinates them into new forms or structures that, in turn, can become the content -what is acted upon- in future acts of abstraction (von Glasersfeld, 1995, p. 69). Battista (1999), in reporting his 3D cube arrays' study, suggested that besides von Glasersfeld's (1995) list of mental mechanisms that includes abstraction and reflection mechanisms there are three additional mechanisms that are fundamental to understanding students' reasoning. They are *spatial structuring, mental models, and schemes* (Battista, 1999; Battista & Clements 1996). *Spatial structuring* is the mental act of constructing an organization or form for an object or set of objects. It determines an object's nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. *Mental models* are nonverbal recall-of-experience-like mental versions of situations; they have structures isomorphic to the perceived structures of situations they represent (Battista, 1999, p. 418; 1994). Mental models consist of integrated sets of abstractions that are activated to interpret and reason about situations that one is dealing with in action or thought. A *scheme* is an organized sequence of actions or operations that has been abstracted from experience and can be applied in response to similar circumstances. It consists of a mechanism for recognizing a situation, a mental model that is activated to interpret actions within the situation, and a set of expectations (usually embedded in the behaviour of the model) about the possible results of those actions (Battista, 1999). Meaningful learning occurs as students make adoptions to their current cognitive structures as a result of their reflection on an experience (Steffe, 1988; Battista, 1999). An accommodation is triggered by a *perturbation* which is described as a disturbance in mental equilibrium caused by an unexpected result or a realization that something is missed or does not work (von Glasersfeld, 1995, p. 67). Perturbation arises when students interact with other individuals or with the physical world (Battista, 1999). von Glasersfeld (1995) made a clear distinction between teaching and training stating that, "From a educator point of view one of the most important features of radical constructivism is the sharp distinction it draws between teaching and training. The first aims at generating understanding, the second at competent performance" (p. xvi). Further, von Glasersfeld in referring to learning mathematics stated that "To know mathematics is to know how and why one operates in specific ways and not in others, how and why the results one obtains are derived from the operations one carries out" (p.xvi).

The Classroom Sessions

First Session**:** Through the classroom problem solving sessions using the seven pieces tangram set, the set was introduced to the students by providing each of them with a colored plastic made model of the seven pieces tangram. The students were instructed that these seven pieces together can be manipulated through motions of translation, rotation and/or reflection to formulate other geometric shapes without overlapping, where the area of each of the resultant shapes made by the seven pieces, is invariant. The students were encouraged to use these seven pieces collectively to form new shapes of equal area. Each of which must be a simply polygon. The instructors explained the concepts of a simple polygon as opposed to a non simple polygon. The students were curious to know the difference and for that the instructors elaborated in saying that a simply polygon is any polygon in which each vertex is created by only two external sides. That is a vertex can have no more than two sides passing through it. This emphasis on the concept of simple polygon is essential since the concept of non simple polygon is rarely included in the middle/high school curriculum. Further, the concept of convex and non convex was also introduced.

Second Session*:* The students started forming as many shapes as they can within the instructions given above using the plastic tangram pieces. They were encouraged to ask questions and were advised to trace and make copies of their constructions. As the students being familiar with the Geometer's Sketchpad (GSP) they were encouraged to use GSP to construct digital images for their shape formations based on the seven pieces tangram.

Third Session: This session was a continuation on the second session but through the computer lab applying GSP. During the computer lab session, the students were challenged with the following question: ***"Given a simple polygonal region and assume it is dissected into a finite number of sub-regions, then what is the maximum number of sides for a simple polygonal region that can be constructed using all sub-regions? Make a conjecture."*** The students then were allowed to take the tangram pieces home for further shape construction, refinement, and show-and-tell opportunity at the following class session. Further, the students were told that they were about to introduce a *new mathematical proposition* should their answer be verifiable. At the tail of the session, one student has asked: "so how far we can go with the number of sides of the new tangram?" The instructor replied: "you may go as far as you see it possible; as a hint, the maximum number of sides possible for a simple polygon is more than twenty!"

Students' Uses of Radical Constructivist Paradigm

Due to the space restriction, two case studies are presented.

Case Study 1: Student, Lee, using the tangram pieces, she has come up with an answer to the question: "*What is the maximum number of sides for a simple polygonal region that can be constructed using all the sub-regions? Make a conjecture as an answer to this question.*" Through a series of constructions started with a square, rectangle, right triangle, parallelogram, trapezoid, pentagon, hexagon, heptagon, octagon, nonagon, decagon, hendecagon, dodecagon, but then Lee emerged with an answer to the question above by stating: "shape with maximum number of sides that can be produced with tangram set is 23 sided polygon." Lee also stated: "the sum of the number of sides of all 7 pieces tangram is equal to the number of sides of the shape that contains the maximum number of sides that can be possibly created using these pieces". She further stated: "Conjecture: **Several shapes can be combined to form a simple, closed shape with maximum number of sides n, where n is the sum of all the sides contained by the sub-shapes altogether**" (Lee's bolding). Lee then offered the figures shown below in Figure 3 a & b.

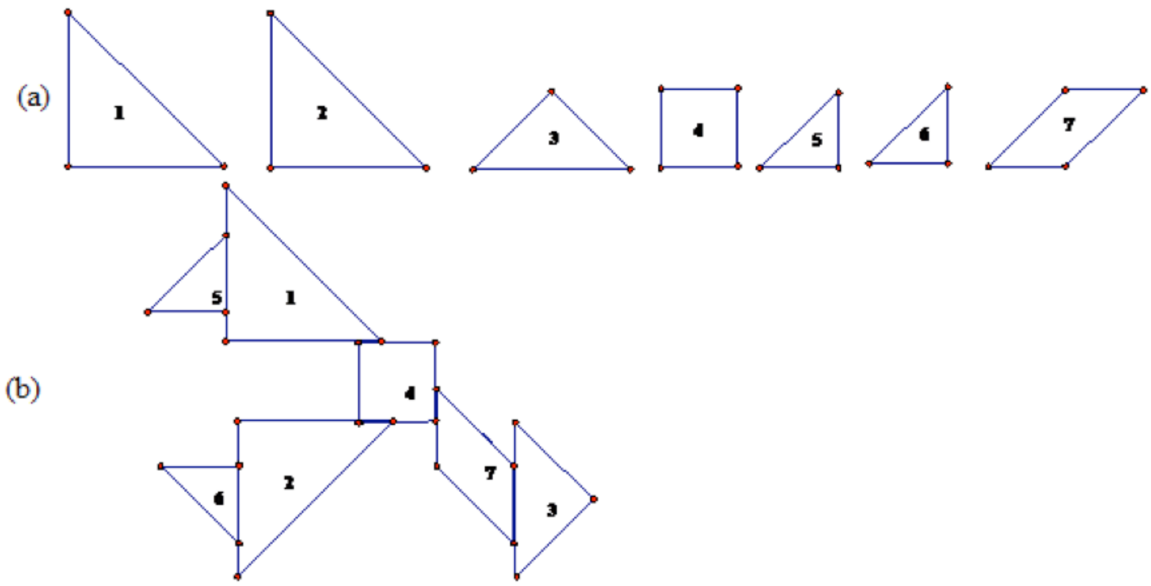
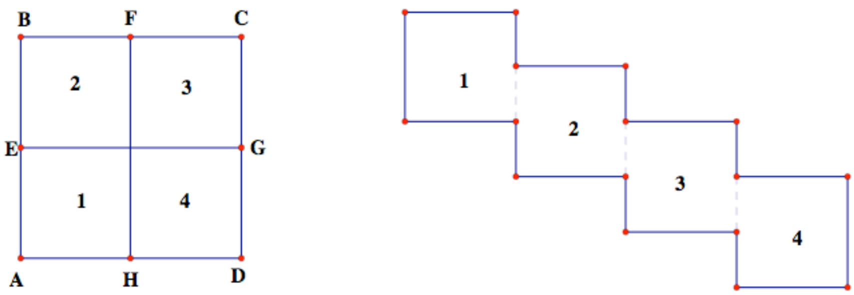


Figure 3: Lee's conjecture

Case Study 2

Student, Brooks, came up with an interesting method for his conjecture in stating that, "for a given square ABCD when dissected into four congruent squares symbolized as 1, 2, 3 and 4 as shown in Figure 4a, the four squares can be rearranged into the shape shown in Figure 4b, and hence, in his words: "*The highest sided polygon is equal to the cumulative number of sides of all of the individual shapes*." Clearly, Brooks meant by his statement that the ultimate created shape using the resultant pieces by dissecting the given square has to have its number of sides to be equal to the total number of the sides of all resultant pieces due to the dissection process. Brooks did not further elaborate on the case when the square ABCD contains the 7 pieces tangram set. Nonetheless, his spatial structuring presents an elegant answer to the question presented above.



(a) **(b)**

Figure 4: Brooks Conjecture

Epilogue

As a reflection on the two cases, out of several cases, within the experimental observations reported, it would seem clear that the von Glasersfeld's radical constructivism theory has been present throughout the students' work on the tangram 7-pieces set activities. Our interpretation of the student, Lee, is that: Lee's processes described in Case Study 1 above (Figure 3 a & b) was directly exemplifying Battista (1999) and Battista & Clements' (1996) interpretation of *spatial structuring* concepts. Evidently, her mental acts of constructing an organization or a form for an object (or set of objects) has been by determining the object's nature or shape, identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. Further, Battista (1999) indicated that material or entity is said to have reached *interiorized level* whenever it has been disembodied from its original perceptual context and it can be freely operated on in imagination, including being "projected" into other perceptual material and utilized in novel situations (p. 418) – Lee was virtually acting along this path. The use of *reflective abstraction* through her mental operations, performed on the previously abstracted items (the 7-pieces shown in Figure 3a), seems to be her building elements that she coordinated into new forms or structures (von Glasersfeld, 1995, pp. 69-70). Then, she coordinated these abstracted pieces or elements to come to make the new form or resultant shape, the 23 sided figure, depicted in Figure 3b. Her acts on the 7-pieces were clearly and directly performed exemplifying Battista's (1995) notion of *interiorized level of abstraction* (p. 418). For the question: "*What is the maximum number of sides you can make for your constructed polygons? Make a conjecture as an answer to this question*." Brooks has also used the notion of *spatial structuring* applied on his pieces, 1, 2, 3, and 4 shown in Figure 4a; Brooks used these perceptual pieces to form the newly created shape

shown in Figure 4b. Brooks, in dealing with his 4 squares, seems to have reached "*interiorized level*" of abstraction in disembodied them from "their original perceptual context". Brooks then has freely operated on the 4 squares "in imagination, including being "projected" into other perceptual material and utilized in novel situations" (Battista, 1999, p. 418). It would seem evident that Brooks has been using *reflective abstraction* through his mental operations, performed on the previously abstracted items along the von Glasersfeld theory (1995, p. 69). Finally, the conjecture that is the crux of these mathematical activities is stated below:
For a given polygonal region, when dissected into a 'k' number of sub-regions, 1, 2, 3, ….., k, with the corresponding number of sides $n_1, n_2, n_3, ..., n_k$, the simple polygonal region with the maximum number of sides that can possibly be constructed by all these sub-regions is $= n_1 + n_2 + n_3 + ... + n_k$.

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