

On Economic Interpretation of Lagrange Multipliers

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Abstract

Lagrange multipliers play a standard role in constraint extrema problems of functions of more variables. In teaching of engineering mathematics they are readily presented as quantities of formal type in the algorithm for finding of constraint extrema. The paper points to important interpretations Lagrange multipliers in optimization tasks in economics

Introduction

The *Lagrange multipliers method* is one of methods for solving constrained extrema problems. Instead of rigorous presentation we point to the rationale of this method. Recall that for a function f of n variables the necessary condition for local extrema is that at the point of extrema all partial derivatives (supposing they exist) must be zero. There are therefore n equations in n unknowns (the x 's), that may be solved to find the potential extrema point (called *critical point*). When the x 's are constrained, there is (at least one) additional equation (constraint) but no additional variables, so that the set of equations is overdetermined. Hence the method introduces an additional variable (the Lagrange multiplier), that enables to solve the problem. More specifically (we may restrict to finding of maxima), suppose we wish to find values x_1, \dots, x_n maximizing

$$y = f(x_1, \dots, x_n)$$

subject to a constraint that permits only some values of the x 's. That constraint is expressed in the form

$$g(x_1, \dots, x_n).$$

The Lagrange multipliers method is based on setting up the new function (the *Lagrange function*)

$$L(x_1, \dots, x_n, \lambda) = f(x_1, \dots, x_n) + \lambda g(x_1, \dots, x_n), \quad (1)$$

where λ is an additional variable called the *Lagrange multiplier*. From (1) the conditions for a critical point are

$$\begin{aligned} L'_{x_1} &= f'_{x_1} + \lambda g'_{x_1} \\ &\vdots \\ L'_{x_n} &= f'_{x_n} + \lambda g'_{x_n} \\ L'_\lambda &= g(x_1, \dots, x_n), \end{aligned} \quad (2)$$

where the symbols L', g' are to denote partial derivatives with respect to the variables listed in the indices. Of course, equations (2) are only necessary conditions for a local maximum. To confirm that the calculated result is indeed a local maximum second order conditions must be verified. Practically, in all current economic problems there is on economic grounds only a single local maximum.

In a standard course of engineering mathematics the Lagrange multiplier is usually presented as a clever mathematical tool („trick“) to reach the wanted solution. There is no large spectrum of sensible examples (mostly a limited number of simple “well-tried” school examples) to show convincingly the power of the method. Economic interpretation of the Lagrangian multiplier provides a strong stimulus to strengthen its importance. This will be central to our next considerations.

Economic interpretations

In the sequel we will examine two useful interpretations $1^0, 2^0$ of the Lagrange multipliers.

1^0 Rearrange the first n equations in (2) as

$$\frac{f'_{x_1}}{-g'_{x_1}} = \dots = \frac{f'_{x_n}}{-g'_{x_n}} = \lambda. \quad (3)$$

Equations(3) say that at maximum point the ratio of f'_{x_i} to g'_{x_i} is the same for every x_i and moreover it equals λ . The numerators f'_{x_i} give the **marginal contribution** (or **benefit**) of each x_i to the function f to be maximized, in other words they give the approximate change in f due to a one unit change in x_i . Similarly, the denominators have a marginal cost interpretation, namely, $-g'_{x_i}$ gives the **marginal cost** of using x_i (or marginal “taking” from g), in other words the approximate change in g due to a unit change in x_i . In the light of this we may summarize, that λ is the *common benefit-cost ratio* for all the x_i 's, i.e.

$$\lambda = \frac{\text{marginal benefit of } x_i}{\text{marginal cost of } y_i} = \frac{f'_{x_i}}{-g'_{x_i}}. \quad (4)$$

Example A farmer has a given length of fence, F , and wishes to enclose the largest possible rectangular area. The question is about the shape of this area. To solve it, let x, y be lengths of sides of the rectangle. The problem is to find x and y maximizing the area $S(x, y) = xy$ of the field, subject to the condition (constraint) that the perimeter is fixed at $F = 2x + 2y$. This is obviously a problem in constraint maximization. We put $f(x, y) = S(x, y)$, $g(x, y) = F - 2x - 2y$ and set up the Lagrange function (1)

$$L(x, y, \lambda) = xy + \lambda(F - 2x - 2y). \quad (5)$$

Conditions (2) are $L'_x = y - 2\lambda, L'_y = x - 2\lambda, L'_\lambda = F - 2x - 2y$. These three equations must be solved. The first two equations give $x = y = 2\lambda$, i.e. x must be equal to y and due to (5) they should be chosen so that the ratio of marginal benefits to marginal cost is the same for both variables. The benefit (in terms of area) of one more unit of x is due to (4) given by $S'_x = y$, (area is increased by y), and the marginal cost (in terms of perimeter) is $-g'_x = 2$ (from the available perimeter is taken 2 for both variables are increased by the same length 1). As mentioned above, the conditions (4) state that this ratio must be equal for each of the variables. Completing the solution we get $x = y = \frac{F}{4}, \lambda = \frac{F}{8}$. Now let us discuss the

interpretation of λ . If the farmer wants to know, how much more field could be enclosed by adding an extra unit of the length of fence, the Lagrange multiplier provides the answer $\frac{F}{8}$ (approximately), i.e. the present perimeter should be divided by 8. For instance, let 400 be

a current perimeter of the fence. With a view to our solution, the optimal field will be a square with sides of lengths $\frac{F}{4} = 100$ and the enclosed area will be 10 000 square units. Now if

perimeter were enlarged by one unit, the value $\lambda = \frac{F}{8} = \frac{400}{8} = 50$ estimates the increase of the total area. Calculating the “exact” increase of the total area, we get: the perimeter is now 401, each side of the square will be $\frac{401}{4}$, the total area of the field is $(\frac{401}{4})^2 = 10050,06$

square units. Hence, the prediction of 50 square units given by the Lagrange multiplier proves to be sufficiently close.

2^o Rewrite the condition $g(x, y) = 0$ in (1) as $c(x, y) = k$, where k is a parameter. Then for the partial derivative of the Lagrange function with respect to k we get $L'_k = -\lambda$. From the interpretation of a partial derivative we conclude, that the value $-\lambda$ states the approximate change in L (and also f) due to a unit change of k . Hence the value $-\lambda$ of the multiplier shows the approximate change that occurs in f at the point of its maxima in response to the change of k by one in the condition $c(x, y) = k$. Since usually $c(x, y) = k$ means economic restrictions imposed (budget, cost, production limitation), the value of multiplier indicates so called the **opportunity cost** (of this constraint). If we could reduce the restriction (i.e. increase k) then the extra cost is $-\lambda$. If we are able to realize an extra unit of output under the cost less than $-\lambda$, then it represents the benefit due to the increase of the value at the point of maxima. Clearly to the economic decision maker such information on opportunity costs is of considerable importance.

Example The profit for some firm is given by $PR(x, y) = -100 + 80x - 0,1x^2 + 100y - 0,2y^2$, where x, y represent the levels of output of two products produced by the firm. Let us further assume that the firm knows its maximum combined feasible production to be 325. It represents the constraint $x + y = 325$. Putting $g(x, y) = x + y - 325 = 0$ we set up the Lagrange function $L(x, y, \lambda) = -100 + 80x - 0,1x^2 + 100y - 0,2y^2 + \lambda(x + y - 325)$. Applying Lagrange multipliers method we get the solution $x = 183,335, y = 141,667, \lambda = -43,333$ with the corresponding value of the profit $PR(183,335; 141,667) = 21358,420$. Now we reduce the restriction altering the constraint equation to $x + y = 326$. Finding the new solution as before we have $x = 184, y = 142, PR(184, 142) = 21401,6$. We see that the increase in profit brought about by increasing the constraint restriction by 1 unit has been 43,18 - approximately the same as the value $-\lambda$ that we derived in the original formulation. It indicates that the additional increase of labour and capital in order to increase the production has the opportunity cost approximately 43,3.

Conclusion

In instructing of engineering mathematics the Lagrange multipliers method is mostly applied in cases when the constraint condition $g(x, y) = 0$ can not be uniquely expressed explicitly as the function $y = f(x)$ or $x = h(y)$. When solving constraint extrema problems in economics the bulk of constraint conditions may be expressed explicitly, so the reason to use the Lagrange multipliers method would seem to be too sophisticated regardless of its theoretical aspects. With a view to the crucial importance of the economic interpretations of Lagrange multipliers is the use of the method primarily preferred. Concrete applications of the presented interpretation principle may be developed in many economic processes. More deeper study on the role of the Lagrange multipliers in optimization tasks may be found in Rockafellar (1993).

References

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