Isomorphic Visualization and Understanding of the Commutativity of Multiplication: from multiplication of whole numbers to multiplication of fractions

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In 2010, I wrote a paper on the developments of visualization, functional materials and actions in teaching mathematics. This paper was a historical survey, but in the recent paper we put emphases on practical issues, taking from the case of the commutativity of multiplication a model for the isomorphic type of visualization we have developed.

From historical view to practical issues on the development of using visualization

In a previous paper of 2010, a discussion was given on the type of visualization used in Finnish mathematics textbooks in the 19th and 20th centuries (Malaty 2010). In the recent work, we are moving to put emphasis on practical issues and present an example, through which we can explore some of the main roles visualization can have in teaching mathematics. Developing mathematics teaching approaches has been always one of my main interests, and working in teacher education and mathematical clubs, in Finland, has given me a chance to develop and test my teaching approaches. One of the elements used in these approaches is visualization (Malaty 2006a, Malaty 2006b, Malaty 2006c, Malaty 1996). Visualization has been for us a facilitator to give students a chance to understand and discover mathematical concepts and relations, and as well a tool to demonstrate, solve and pose mathematical problems. The example we are going to present is related to the teaching of the commutativity of multiplication.

Common mistakes in the visualization of the commutativity of multiplication

Fig.1 represents one of our textbooks' visualization (Haapaniemi et al. 2002, 126). At first, we can notice that the figure contains unneeded elements, and this makes it far from the simplicity of iconic visualization. Such mistake is common in textbooks, but the main problem, we see, is the use of visualization to make ungrounded generalizations. This is as well the case in our example. The textbook uses the given figure (Fig.1) to present two statements $5\cdot 2=10$ and $2\cdot 5=10$. But these statements are not justified by the figures presented, but by the multiplication tables. Visualization role here is a modest one, and near to be only decorative. This is still the case, when textbooks provide more iconic visualization. In Fig.2, of the same textbook, and in the same page, two statements, $3\cdot 2=6$ and $2\cdot 3=6$, are introduced and justified by multiplication tables and not upon visualization.

Fig. 1. Complicated visualization

Fig. 2. Iconic visualization

Textbooks' effect is rematkable on what happanes in the classroom, including the use of textbooks' visualization. But, other types of mistakes have been observed in classroom teaching. For the commutativity of multiplication, not rare to see on the boards a type of mechanical performance like the one of Fig.3. While the numbers included in the statement $2\cdot 3=3\cdot 2$ are visualized (Fig.3), no attention is given to the meaning of the multiplication operation. Numbers are visualized, as if for summands of a sum and not for multiplier and multiplicand of a product.

$$2\cdot 3=3\cdot 2$$

Fig.3. Mechanical performance

Iconic, dynamic 2-dimensional visualization

In building visualization for mathematical relations, we have developed iconic dynamic types of visualization to get isomorphism with the relation we visualize. For the commutativity of multiplication, our types of visualization are 2-dimentional ones. At the beginning, and for young children, we use perpendicular dimensions, but at relevant stage we ask students to investigate the possibility of using oblique dimensions.

Visualization of the commutativity of multiplication of whole numbers

This I start by writing on the board the expression $2\cdot 3$. After that, I draw only one row of three circles, as in Fig. 4. Then I ask children:

Fig.4. One row of three circles

How many times three red circles I have drawn? – Only one time. After getting this answer, I draw an arrow as in Fig. 5, and then continue: *Right. But, here above is written* $2\cdot 3$ *, what I have to do?* – Draw another row. *Here?* I add a new arrow as in Fig.6, and then I draw the new row of three red circles as in Fig.7

Fig.5. Adding an arrow to mean one time of three circles

Fig.6. Adding another arrow

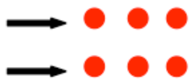

Fig.7. Drawing the other row of three circles

The 2-dimentional iconic visualization (Fig.7) has been built in a dynamic way to get a figure isomorphic to the expression $2\cdot 3$. After that, we can continue our work with children in different ways, taking in mind the level of the group. A challenging and interesting one is to continue as follows: *But, this figure* (Fig.7) *is also representing the expression* $3\cdot 2$ (where I write numeral 2 in red and numeral 3 in other color like green), *who can tell me how this is possible?* If my last question does not help, I can use another approach like the one of the next dialog. *How many red circles you can see in one column?* (I draw in green vertical arrow as in Fig. 8, and then also in green a vertical segment as in Fig.9) – Two. *How many times such 'two circles' we have?* - Three. Yes, *who can draw more arrows to show that we have'3 times 2 circles'?* When a student gets a figure like that of Fig.10,

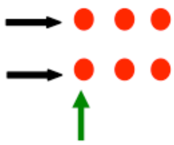

Fig.8. Adding in green vertical arrow

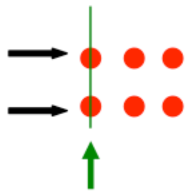

Fig.9. Adding in green vertical segment

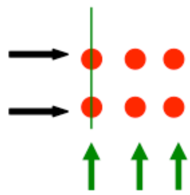

Fig.10. Drawing more vertical arrows

we continue: Great. *'3 times 2'*, this expression I want to write here; *which color is relevant for writing numeral 3?* – Green. Right. *What about the color relevant for numeral 2?* – Red. *Why not green or other color?* – Because, here, 2 is the number of red circles. At this moment, I write the expression $3\cdot 2$ in the same row of the expression $2\cdot 3$, which has been already written on the top of the board, and leave a space between the two expressions to draw a box in a form of a rectangle, as in Fig.11. In addition, I put three large cards over the open sentence, two for inequality relations and the third for the relation of equality, as in Fig.12. Then, I ask children to have a time to think about the relevant card to change the open sentence into a true statement? *Take your time, and raise your hand when you are ready to tell me, which card is relevant and where to put it?*

$2\cdot 3 \ \boxed{} \ 3\cdot 2$

Fig.11. Open sentence

$\boxed{} \ \boxed{} \ \boxed{}$?

$2\cdot 3 \ \boxed{} \ 3\cdot 2$

Fig.12. Adding three large cards of relations

Two meanings of dynamic visualization, and isomorphism

Getting Fig.10 for the commutativity of multiplication was an outcome of a process and not given as such, at once. For this reason we call such

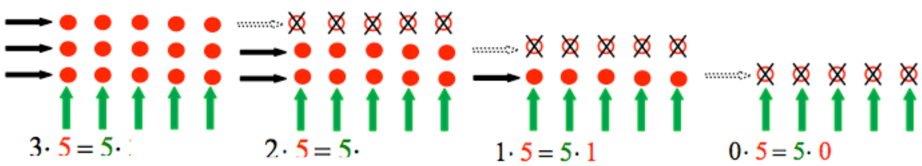

Fig.13. From $3\cdot 5 = 5\cdot 3$ **, to** $2\cdot 5 = 5\cdot 2$ **then** $1\cdot 5 = 5\cdot 1$ **and Finally** $0\cdot 5 = 5\cdot 0$

visualization a dynamic visualization. Nevertheless, the main reason of giving this name is the possibility of modifying such figure to present other cases, including the general case. For instance, Fig.10 can be used by students to show that the commutativity of multiplication does exist for any two whole numbers. Among others, students can modify Fig.10 to show that, $2\cdot 5=5\cdot 2$, $3\cdot 5=5\cdot 3$, $3\cdot 7=7\cdot 3$, $4\cdot 7=7\cdot 4$, $5\cdot 7=7\cdot 5$, $5\cdot 10=10\cdot 5$, $5\cdot 100=100\cdot 5$, $5\cdot 104=104\cdot 5$ and $7\cdot 104=104\cdot 7$. Not only such cases can be discussed, but students can visualize a statement like $17\cdot 104=104\cdot 17$ before they learn how to perform in any of this statement's sides. This is an evidence of the isomorphism of our type of visualization with the relation it visualizes. In addition, we can modify this 2-dimensional iconic dynamic visualization to allow us to visualize special cases, in particular $0\cdot 5=5\cdot 0$ and $0\cdot 0=0\cdot 0$, as Fig.13 and Fig. 14 show.

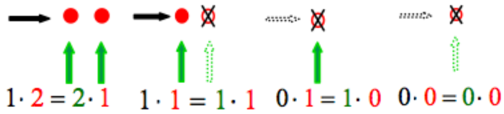

Fig.14. From $1\cdot 2=2\cdot 1$, **to** $0\cdot 0=0\cdot 0$

Isomorphic visualization and generalization

The most important is that, our visualization can be used for the general case of the commutativity of multiplication for whole numbers. The general case can be presented by the statement; $n\,a = a\,n$, where n and a are whole numbers. In Fig.15, because the number of red circles in a row is a and we have n rows, the total number of circles is na. As we have n rows, in a column we must have n circles. But, the number of such columns is a, because in a row we have a circles. Thus, the total number of circles is an. Now, as Fig.15 gives us the chance to visualize the general case, this figure can be modified to visualize the special cases $1\cdot 5=5\cdot 1$ and $0\cdot 5=5\cdot 0$, presented above (Fig.13), in the general form; $1\cdot a=a\cdot 1$, $0\cdot a=a\cdot 0$.

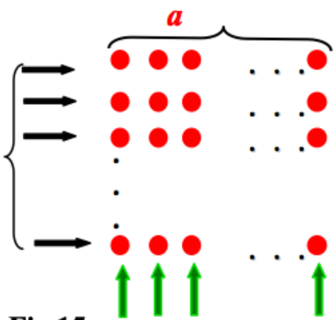

Fig.15.

Visualization's isomorphism with the algebraic proof and the need for visualization in writing proofs

$$\begin{aligned} n\,a &= n\,(1+1+1+\ldots+1 \quad [a\ terms]) \\ &= n+n+n+\ldots+n \quad [a\ terms] \\ &= a\,n \end{aligned}$$

Fig.16. A proof of distributivity

In Fig.16 we give a simple proof for the commutativity of multiplication for whole numbers. This proof is built on the distributivity of multiplication over addition for whole numbers. And, this distributivity can be proved easily, upon the associativity of addition for whole numbers. For this chain of proofs, we need to have this associativity as an axiom. In addition, both associativity and distributivity properties can be easily visualized. Regards the isomorphism of algebraic proof (Fig.16) with the visualization of the general case (Fig.15), we can notice that ones in the first line of the proof (Fig.16) match the red circles in the first row of Fig.15. In addition, n in the first line of the proof (Fig.16) corresponds the rows' number in Fig. 15. In the second line of the proof (Fig.16), n corresponds the number of circles in a column (Fig.15), while in the algebraic proof it is the result of multiplying 1's by n according to the distributivity of multiplication over addition. Without using distributivity, we can rely on the concept of multiplication to prove that:

$$\begin{aligned} & n\,(1+1+1+\ldots+1 \quad [a\ terms]) \\ = \; & 1+1+1+\ldots+1 \quad [a\ terms]+ \\ & 1+1+1+\ldots+1 \quad [a\ terms]+ \\ & 1+1+1+\ldots+1 \quad [a\ terms]+ \\ & \vdots \\ & 1+1+1+\ldots+1 \quad [a\ terms] \\ = \; & n+n+n+\ldots+n \quad [a\ terms]. \end{aligned}$$

Fig.17. Proof based on the concept of multiplication

$n\,(1+1+1+\ldots+1 \quad [a\ terms]) = n+n+n+\ldots+n\,[a\ terms]$ (Fig.17).

This last proof (Fig.17) shows how our visualization for the general case of the commutativity of multiplication for whole numbers (Fig.15) is not different than this proof. The n series in the proof (Fig.17) correspond the n rows in our visualization (Fig.15), and the ones in the proof (Fig.17) correspond the red circles (Fig.15).

This shows the need to develop visualization to become as much as possible isomorphic to the mathematical entity visualized. For young children, this gives them a chance to think mathematically and be ready to move to more abstract level, like the algebraic level. When we look at the last proof, we can notice that, bringing n's at the last line (Fig.17) is done visually from the ones of the columns over these n's, and this what we are doing in an isomorphic way in our visualization (Fig.15). Getting these n's of the proof (Fig.17) algebraically is possible, and this can make the algebraic proof rather closed to our visualization of the general case (Fig.15). For space reason, we leave this proof outside this paper. But, we have to mention that such proof can show how visualization is still needed in writing this proof. This means that visualization is not only needed for young children, but as well in studying mathematics and making mathematics.

Isomorphic visualization, multiplication by a fraction and unit fraction

The isomorphism between our 2-dimensional dynamic iconic visualization and the commutativity of multiplication for whole numbers is clearly strong, and therefore we can use it in a modified form to visualize the commutativity of multiplication of fractions. Fractions here are proper fractions, those less than one, and improper fractions, those equal or greater than one. Thus, whole numbers are also fractions. When we start to work with multiplication by fractions, different than whole numbers, we need to develop our reading of a product. For instance reading $\frac{1}{3}\cdot 6$ as "one third times 6" is difficult to understand, and therefore we need to modify our reading style as we in fact making enlargement for the concept of multiplication. Thus, instead of saying three times six, two times six, one times six and "one third 'times' six"; it is now a time to say three sixes, or three of sixes; two sixes, or two of sixes; one six; one third six or "one third 'of' six". Reading "one third of six" brings to light the relation between multiplication by a fraction and division by the reciprocal of this fraction. With this reading style, we can easy get the value of the product. For instance, one third of six is two as it is the quotient we get in dividing 6 by 3. Here to remember that in algebra we do not say 2 times a, but $2a$. One thing more, related to fractions and fractions multiplication, we have to here mention. This is the concept of unit fraction, which is a fraction with 1 as nominator. This concept wasn't known in Finnish schools, but our work has made a positive change towards understanding and using of it. For instance, in multiplying $\frac{3}{8}$ by 2 we underline the fact that we need to multiply 3 by 2, exactly as multiplying 3 ones by 2. The only difference is the unit of the product. In multiplying three eighths by two the product is not 6 ones, but 6 eighths, as the unit of the multiplicand is not of ones, but of eighths. The idea of "unit" has a wider role in our work. Regards arithmetic and decimal system, place value has a special meaning, as each digit's value depends on the unit of the place. Our experience shows that understanding of the unit idea brings meaning to children for their performance of skills and makes this performance easier. And, both achievements bring motivation for the study of mathematics.

Commutativity and visualization of the multiplication of a whole number by a unit fraction

Multiplying a whole number by $\frac{1}{1}$, as improper unit fraction, is possible to discuss. But, this is an easy and trivial case, and therefore we move to discuss a more substantial case, and take the expression $\frac{1}{5}\cdot 3$, where 3 is multiplied by a unit fraction $\frac{1}{5}$.

Fig.18. Drawing a rectangle

To get isomorphic visualization to the expression $\frac{1}{5}\cdot 3$, we start by writing this expression on the board, where I use a color, like red here, in writing the multiplicand.

Then I ask students: *What is the multiplicand we have in this expression?* – Three. *We now shall try to represent the multiplicand 3 by rectangles, are you ready?* –Yes. Which color, I have to use in drawing the rectangles? – Red. Then I draw a rectangle as in Fig.18, and ask students: *How many rectangles I have drawn?* – One. Yes, *how many of such rectangles I need to draw more?* – Two. Right, then I draw two more rectangles as in Fig.19. *Who can read the expression, written on the top of the board?* – "One fifth of three". Great, *who can use blue color and draw a straight line to find and present to us one fifth of three?* Such approach or a modified one has to lead to get a figure like the one of Fig.20. Then we again give students a challenge by saying: but, this figure in fact shows that $\frac{1}{5}\cdot 3 = 3\cdot\frac{1}{5}$. *Who can show us that this statement is true?* This discussion has to lead us to a figure with three arrows as in Fig.21.

Fig.19. Visualization of 3

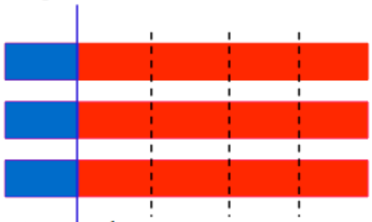

Fig.20. $\frac{1}{5}\cdot 3$

The color used in writing the fraction $\frac{1}{5}$ is blue, and this is because we have agreed with students to color the part of the three rectangles, which represents one fifth of three, by the same color we give to the line we use in cutting the rectangles. We can make a story about the effect of the color of the line used. Speaking about using of stories in teaching mathematics, in multiplying 3 by one fifth we can use a story like 5 children in a party sharing equally three cakes of different taste. We can use also a type of functional materials to help those in need of working on hands, where they can cut by themselves strips of red colored papers.

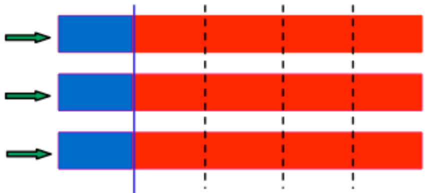

Fig.21. $\frac{1}{5}\cdot 3 = 3\cdot\frac{1}{5}$

Commutativity and visualization of the multiplication of a whole number by a fraction different than unit fraction

Let us here start with a proper fraction $\frac{2}{5}$ as multiplier and use the same multiplicand of the last case, i.e. 3. After dealing with the case of $\frac{1}{5}\cdot 3 = 3\cdot\frac{1}{5}$, it is not difficult to visualize $\frac{2}{5}\cdot 3 = 3\cdot\frac{2}{5}$, and this can be given to students as a problem to solve. Fig.22 presents such visualization. Here, we do not use colors in writing the statement. As we need to free students from using such colors. About such coloring use, on one hand, we can go back to use such type of coloring, when we find it necessary to recall ideas, and on the other hand colors can work in an imaginary form for both students and teachers in their discussions. In our case, when we do not use colors in writing statements, we can visualize the expression 'two fifth of three' using two colors, instead of one. In Fig.22 we have colored "one fifth of three" in blue and the other "one fifth of three" in yellow. The other choice is to color both in one color, like blue. Here to notice that, Fig.22 is modified from Fig.21, and this show that we can continue this process, to go from Fig.21 to any case of the multiplication of a whole number by a fraction, proper or improper. In addition, we can modify Fig.21 to visualize the general case of multiplying a whole number by a fraction, i.e. we can use a visualization to show that $\frac{m}{n}\cdot a = a\cdot\frac{m}{n}$, where *m, n* and a are whole numbers and $n \neq 0$. We can also investigate the isomorphism of the algebraic proof of this statement and the figure visualizes it.

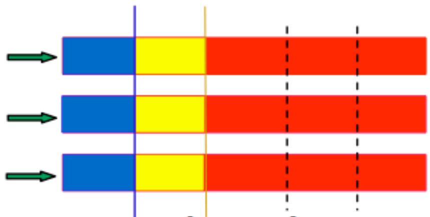

Fig.22. $\frac{2}{5}\cdot 3 = 3\cdot\frac{2}{5}$

Commutativity and visualization of the multiplication of a fraction by a fraction

Let us start with the visualization of the commutativity of multiplication of a unit fraction by a unit fraction, like $\frac{1}{2}\cdot\frac{1}{3}=\frac{1}{3}\cdot\frac{1}{2}$. At first we visualize a whole by a rectangle (Fig.23), and then draw two segments to divide the rectangle into 3 congruent rectangles and color one in blue to visualize $\frac{1}{3}$ (Fig.24). To get 'half of one third' we draw a horizontal segment, in red to divide the blue rectangle into two congruent rectangles and color one in red, here the lowest. This 'half of one third' (Fig.25) is obviously 'one third of a half', as we can imagine, or here draw in a similar way a horizontal segment in the whole rectangle to divide it into two congruent rectangles and color the lowest in red to visualize half (Fig.26), then draw a vertical segment in blue to get 'one third of this half' (Fig.27).

Fig.23. Visualization of a whole

Fig.24. Visualization of $\frac{1}{3}$

Fig.25. $\frac{1}{2}\cdot\frac{1}{3}$

Fig.26. Visualization of a $\frac{1}{2}$

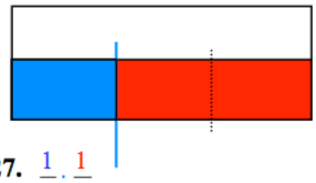

Fig.27. $\frac{1}{3}\cdot\frac{1}{2}$

Reflections

Last discussed visualization is for the commutativity of the multiplication of two unit fractions, but we can easy modify it for use for any two fractions and for the general case, where improper fractions are included. Algebraic proof can be provided to the statement of general case, and this proof can show how our iconic dynamic visualization is isomorphic to mathematics. Visualization can bring understanding of mathematics, but on the other hand making visualization needs understanding of mathematics. All our figures in this paper are 2-dimensional ones. Using of rectangles is possible to the case of whole numbers, and this use has a closed relation with the proof of rectangle's area statement. In our visualization above, parallelograms can replace rectangles. This is similar to the case of the 2-dimensional visualization for the multiplication of a whole number by another whole number, where we can use oblique dimensions.

References

Haapaniemi et al. 2002. Tuhattaituri, 2a, [Spring textbook in mathematics for Grade 2]. Helsinki: Otava.

Malaty, G. 1996. Joensuu and mathematical thinking. In: *Mathematics for Tomorrow's Young Children*. Dordrecht: Kluwer Academic Publishers, 302-316.

Malaty, G. 2006a. The Role of Visualization in Mathematics Education: Can visualization Promote the Causal Thinking? *Matematicheskoe obrazovaнue: proshloe, nastojashtsheee, budushtsheee,* Moskva, 247-252.

Malaty, G. 2006b. What are the Reasons Behind the Success of Finland in PISA? *Gazette Des Mathématiciens,* 59-66.

Malaty, G. 2006c. Mathematical Clubs: A way to Develop Mathematics Education. *Proceedings of the section Nordic Presentations at ICME-10, July 12, 2004 in Copenhagen (Denmark),* Research Report, 83-90.

Malaty, G. 2010. Developments In Using Visualization, Functional Materials And Actions In Teaching Mathematics: understanding of relations, making generalizations and solving problems (Part I). In: *Mathematical Education in a Context of Changes in Primary School, Conference Proceedings,* pp.18-26. Olomouc: University of Olomouc.