Adjusting the Mathematics Curriculum Into the 21st Century

Classroom Examples

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Abstract

In our efforts to improve the mathematics curriculum and adjust the mathematics education into the 21^{st} century, we have developed a technology based course at a teacher education college. Technology enables us to integrate into the school curriculum topics which were until now only taught in higher math education courses. Raising students' awareness to the mutual relationship between math and technology will broaden the students' view of the nature of mathematics and its applications in the real world.
This paper focuses on one of the topics taught in the course- finding roots of various kinds of equations. We begin with computing square and cubic roots using the intuitive 'trial and error' method followed by Heron's method (100 a.d.). Then we generalize both to first and second order numerical methods which enable to solve even equations which have no analytic solving formulas.
We use the GeoGebra software to obtain the graphs of the functions or to check student's answers which were obtained using calculus. The students define the number of the solutions (if any). They get acquainted with the numerical methods, write their own algorithms, translate them into computer programs and get the solution by using Excel.
We hope that adding new and vital subjects, which are ordinarily absent from the regular school programs (in Israel), using technology and rich learning tasks, will make the shift in mathematics education.

Introduction

The last decades are characterized by rapid changes in mathematics teaching all over the world. Technology changes the way students think and learn (Dori, Barnea, Herschkovitz, Barak, Kaberman, and Sason, 2002) and demands adequate changes in the curriculum as well. "In mathematics instruction programs, technology should be used widely and responsibly, with the goal of enriching students' learning of mathematics (NCTM, 1991).
In our efforts to improve the mathematics curriculum, students' math understanding and adjust the mathematics education into the 21^{st} century, we have developed a technology based course**.** This course is taught to mathematics B.Ed and M.Ed students in a teacher training college.
Technology enables us to integrate into the school curriculum topics which were until now only taught in higher math education courses. Raising students' awareness to the mutual relationship between math and technology will broaden the students' view of the nature of mathematics and its applications in the real world. Lecturing at a teacher education college, we believe that after participating in the course, our students will incorporate its topics into schools.

This paper describes one of the topics taught in the course - finding roots of various kinds of equations. This is one of the issues mathematicians dealt with from the dawn of history, in order to solve various kinds of mathematical problems.
Students are not aware that (according to Abel, Galois and Lie) there does not exist and will never be found a closed formula for solving polynomial equations of an order greater than 4, and for other non algebraic equations such as exponential or trigonometric (Arbel, 2009). Working with advanced software (such as Matlab) one can get solutions to those problems. We Raise students' mathematical curiosity as to how the computer functions. In other words, they learn "the story behind the key" of calculators, graphic calculators and behind computer built in library functions.
During the course the students get acquainted with the mathematical ideas and numerical methods absent from the school curriculum (in Israel). At first the students use calculators, then they write an algorithm and translate it to a computer program using Excel. Realizing the "strength" of computers they construct permanent software that is both efficient and fully automatic.

Computing square and cubic roots

1. <u>The intuitive 'trial and error' method</u>

This method is based on finding two sequences of upper and lower bounds which get closer and closer to the root (until the desired accuracy is reached).
For example:
To compute $\sqrt{5}$ we start by intuitively finding two numbers a, b as lower and upper bounds for $\sqrt{5}$. In our example $2<\sqrt{5}<3$ (first approximation). Knowing that $\sqrt{5}$ is greater than 2 and smaller than 3, let us take 2.5 (the average) as our next intuitive approximation. Now we calculate the value of x^2.If x^2 is smaller than 5 the new approximation is the new lower bound a, otherwise it is the upper bound b. We continue until a is close enough to b. The algorithm is shown in figure 1.

***COMPUTING THE SQUARE ROOT OF S** Trial and Error*
1. input s (positive number)
2. input a, b a< s^0.5 <b
3. <u>while</u> |a = b (to a desired accuracy) <u>do</u>
3.1 x ⟵ (a+b)/2
3.2 if x^2 <s then a ⟵ x
else b ⟵ x
4. print x
5. end

Figure 1

The next stage is translating the algorithm to a computer program (using Excel) –see figure 2 below.

2. <u>Heron's (100 a.d) iterative formula for computing the square root of s (a given positive number)</u>

Heron's method is based on creating a sequence of rectangles, all with area S. In each new rectangle both sides are getting closer to each other. As a limit of the sequence we get a square. The sides of this square are the desired square root of S.

trial and error					Heron	
a	b	x	x2	S	a	b
2	3	2.5	6.25	5	1	5
2	2.5	2.25	5.0625		3	1.666666667
2	2.25	2.125	4.515625		2.333333333	2.142857143
2.125	2.25	2.1875	4.78515625		2.238095238	2.234042553
2.1875	2.25	2.21875	4.922851563		2.236068896	2.236067059
2.21875	2.25	2.234375	4.992431641		2.236067977	2.236067977
2.234375	2.25	2.2421875	5.027404785		2.236067977	2.236067977
2.234375	2.2421875	2.23828125	5.009902954			
2.234375	2.23828125	2.236328125	5.001163483			
2.234375	2.236328125	2.235351563	4.996796608			
2.235351563	2.236328125	2.235839844	4.998979807			
2.235839844	2.236328125	2.236083984	5.000071585			
2.235839844	2.236083984	2.235961914	4.999525681			
2.235961914	2.236083984	2.236022949	4.999798629			
2.236022949	2.236083984	2.236053467	4.999935106			
2.236053467	2.236083984	2.236068726	5.000003346			
2.236053467	2.236068726	2.236061096	4.999969226			
2.236061096	2.236068726	2.236064911	4.999986286			
2.236064911	2.236068726	2.236066818	4.999994816			
2.236066818	2.236068726	2.236067772	4.999999081			
2.236067772	2.236068726	2.236068249	5.000001213			
2.236067772	2.236068249	2.23606801	5.000000147			
2.236067772	2.23606801	2.236067891	4.999999614			
2.236067891	2.23606801	2.236067951	4.99999988			
2.236067951	2.23606801	2.236067981	5.000000014			
2.236067951	2.236067981	2.236067966	4.999999947			
2.236067966	2.236067981	2.236067973	4.99999998			
2.236067973	2.236067981	2.236067977	4.999999997			
2.236067977	2.236067981	2.236067979	5.000000005			
2.236067977	2.236067979	2.236067978	5.000000001			
2.236067977	2.236067978	2.236067977	4.999999999			
2.236067977	2.236067978	2.236067978	5			
2.236067977	2.236067978	2.236067977	4.999999999			
2.236067977	2.236067978	2.236067977	5			

COMPUTING THE SQUARE ROOT OF S –

HERON OF ALEXANDRIA

1. input q (desired no. of correct figures)

2. input s (positive no.)

3. a ← 1

4. b ← s

5. E ← abs(a-b)

6. <u>while</u> E>10^ -q <u>do</u>

6.1 a ← (a+b)/2

6.2 b ← s/a

7. <u>else</u> print a, b

8. end.

Figure 2

In our presentation we will show how the students expand Heron's algorithm to a fully automatic one that is stopped when a desired accuracy of q significant digits is obtained. We will also compute the cubic root similarly.
Another way of calculating the digits of the square root of a given number s, is by finding the roots of the equation x^2-s=0.

Solving equations

In order to solve f(x)=0, in other words to find the real roots of the equation, we look at the function y= f(x) and solve $\begin{cases} y = f(x) \\ y=0 \end{cases}$ (points of intersection with the x axis).

To investigate the given continuous function y=f(x) the students are asked to:

- Plot the graph of the given function (using calculus and/or GeoGebra software). The software is used either for checking students' answers which were obtained using calculus, or to obtain the graphs of the functions.
- Decide the number of zeros (if any).

We will now present two methods that enable to solve (numerically) all kinds of equations, even those which have no analytic solving formulas (exponential, trigonometric or polynomial of a degree greater than 4). These methods are a generalization of both methods discussed previously.

1. Bisection method
For each of the zeros of an increasing function

Choose a relevant interval [a,b] where f(a) <0 and f(b) >0 (switch f(a) with f(b) for a decreasing function). The required root lies between a and b (Cauchy's mean value theorem), precisely where the graph intersects the x axis.
Let x_m= (a+b)/2 be the midpoint of the interval. Compute y= f(x_m)
If y<o take x_m as the new a or else take x_m as the new b.

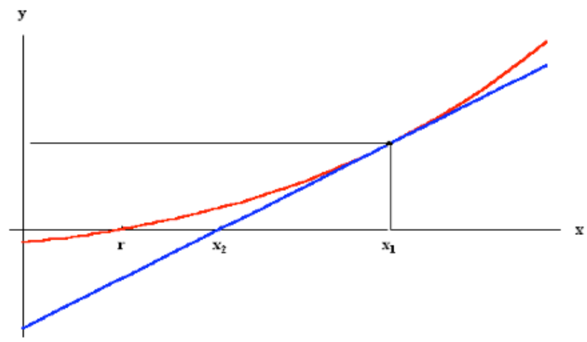
- Continue until the desired accuracy is reached. (Breuer and Zwas, 1993).

The algorithm and the program for calculating the square root:

1. Input s
2. Input a, b, f(a)<0, f(b)> 0
3. $x \leftarrow (a+b)/2$
4. $f(x) \leftarrow x^2 - s$
5. While $f(x)\neq 0$ do
   5.1 if $f(x)>0$ $x \leftarrow b$
   Else $x \leftarrow a$
   5.2 $x \leftarrow (a+b)/2$
   5.3 $f(x) \leftarrow x^2 - s$
6. print x

Another example solving xe^{-x} -0.25=0, will be presented during the presentation.

3. **Newton Raphson Method- using the tangent line** for a differentiable function



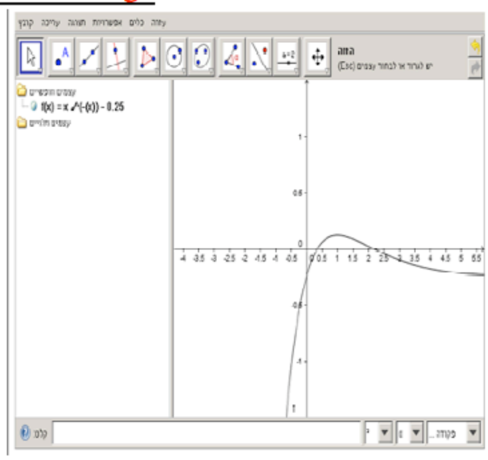
- Choose x_1 to be the first approximation for the root r. $y_1=f(x_1)$.
- Find $f'(x_1)$.
- At (x_1, y_1) calculate the tangent line.
- Find x_2, the point where the tangent line intersects the x axis. x_2 lies closer to r, therefore x_2 is chosen to be the next approximation of r.
- Continue similarly until f(x)=0 (the desired accuracy is obtained).
- For each iteration: Find $y=f'(x_n)$ and compute
  $\boxed{x_{n+1}=x_n-f(x_n)/f'(x_n) \qquad f'(x_n)\neq 0}$ **Newton Raphson** formula (1690).

$\boxed{\begin{array}{l} x_n+1=x_n-f(x_n)/f'(x_n) \\ f'(x_n) \neq 0 \end{array}}$

Newton Raphson Method **$f(x)=xe^{-x}-0.25$**

x	f(x)	f'(x)	
0.5	0.05326533	0.3032653	
0.324360635	-0.01549068	0.4884801	increasing
0.35607263	-0.00059899	0.4510207	
0.357400699	-1.01E-06	0.4494932	
0.357402956	-2.93E-12	0.4494906	
0.357402956	0	0.4494906	
0.357402956	0	0.4494906	

x	f(x)	f'(x)	
2.5	-0.0447875	-0.1231275	
2.136251007	0.002284318	-0.1341876	
2.153274331	2.41E-06	-0.133899	
2.153292364	2.89E-12	-0.1338987	
2.153292364	0	-0.1338987	
2.153292364	0	-0.1338987	decreasing



the algorithm

```
1. a ← x1
2 b ← f(a)
3. c ← f'(a)
4. while f(x)≠0  do
  4.1   print a
  4.2   a ← a-b/c
  4.3   b ← f(a)
  4.4   c ← f'(a)
5. end.
```

One can see:

- When dealing with the same equation xe^{-x}-0.25=0, the Newton Raphson Method (of the second order) gives the solution much quicker than the bisection method (of the first order).
  For x^2-s=0 Newton Raphson's method yields the same formula (and result) as Heron got without using calculus.

During the course, the students have learned many and varied numerical methods taken from different branches of mathematics. Emphasis is given to the mathematical knowledge and to accompanying justifications.
We believe that technological developments make it possible to incorporate selected chapters of this course in high school or even in the upper grades of the elementary school curriculum, by adapting the topics to students' knowledge.

We hope that these topics will be integrated into the curriculum and our students will be the agents who incorporate it into schools.

References

Arbel, B. (2009). *Mathematicians and great events in the history of mathematics*. Mofet institute, Israel. (in Hebrew).

Breuer, S., & Zwas, G.(1993). *Numerical Mathematics. A Laboratory Approach*, Cambridge University Press.

Dori,Y., J., Barnea, N., Herschkovitz, O., Barak, M., Kaberman, Z. and Sason, I. (2002). Teacher Education toward teaching in technology - enriched environment and developing higher - order thinking skills. *Proceedings of the 4th international conference on teacher education: Teacher education as a social mission*, vol.1, pp. 123. Achva college; Israel.

NCTM (1991). *Professional Standards for Teaching Mathematics*. Reston, VA.