Horizontal and Vertical Concept Transitions

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Abstract:
Transfer of concepts, ideas and procedures learned in mathematics to a new and unanticipated situation or domain is one of the biggest challenges for teachers to communicate and for students to learn because it involves high cognitive skills. This study is an attempt to find ways for driving students to generalize and expand mathematical results from one domain to another in a natural way, and to promote that mathematics is not a collection of isolated facts by providing meaningful ways for students to construct, explain, describe, manipulate or predict patterns and regularities associated with a given system of theorems and mathematical behavior. One would wish there were a universal genetic decomposition for generalization and for the abstraction of properties from a given structure and applying it to a new domain. In this study I plan to focus on particular cases of generalizations in calculus and distinguish between two different types of examples.
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Many instructors choose to informally display concept transfer by just drawing the students' attention to the connection between the initial result and the new transferred result so as to make the extension seem natural in retrospect. This helps the learner gain ownership over the mathematical content. One could say that even if the students are not able to deduce the generalized concepts on their own, at least they are convinced that those mathematical facts are the result of a human activity (Streefland, 1991, p. 15). To facilitate this, it helps to view effective learning as a series of processes of horizontal and vertical mathematization that together result in the reinvention of mathematical ideas. Realistic Mathematics Education (RME) is about how mathematics is learned and at the same time about how it should be taught: students must be guided and encouraged to create their own systems or internalize the process of such creations; because by doing that they learn best or even reinvent mathematics. RME distinguishes between two types of "mathematization": horizontal and vertical mathematization. Horizontal mathematization involves set patterns, rules and models that should be learned and applied with given principles. Vertical mathematization, on the other hand, requires flexibility and allows students to shape and manipulate mathematical results. The first type relies primarily on memorization; whereas the second requires higher cognitive skills and activities. Students will develop attitudes to mathematics more in line with those preferred by mathematicians while standard mathematics lectures designed to "get through the material" may force them into rote-learning habits that mathematicians hate.

Unfortunately, many teachers still choose to teach mathematics "as a set of rules of processing or…algorithms" because "it is the way they learned it themselves" (Freudenthal, 1991, p. 3). Those instructors, I believe, are suspected to be the ones who usually tend to overemphasize details and conditions earlier on, at the expense of suppressing mathematical intuition and free "guessing".

In the examples below I distinguish between two types of generalizations:

1. The chain rule for the case of $y = f(u(x))$ is $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$. In the case of higher dimensions it translates into $\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$, when $z = f(x, y)$ and $x = x(t), y = y(t)$.

2. Total differential: $dy = f'(x)dx$ for a function of one variable translates into: $dz = f_x dx + f_y dy$. For the case of a function of two variables. A plausible question would be: why not the average of the last two addends? Specially that the error's upper bound involves an average. $|E| \leq \frac{1}{2} Max(f_{xx}, f_{xy}, f_{yy})$, where the maximum is taken over a certain domain.

3. Taylor theorem as an extension of linearization.

4. Independence of path in the case of line integrals as an expansion of the Fundamental theorem of calculus (into line integrals).

5. Fourier series as an inspiration from Taylor theorem: while Taylor theorem models certain functions as power series, Fourier's theorem models certain periodic functions as trigonometric series.

There is an obvious distinction between two different types of generalizations in the examples above: when a result A is a generalization of a result B, then A could be seen as a special case of B. However when a result D is an abstraction of a result C, then it means that they both share properties. Or that are applied to different fields altogether, or that the properties themselves are extensions of one another. I refer to this distinction by vertical and horizontal transition:

- The chain rule for the case of a function of a single variable can be seen as a special case of the chain rule for a function of several variables; likewise, linearization can be perceived as a special case of Taylors theorem for $n = 1$.
  Moreover, the Fundamental Theorem of Calculus $\int_a^b f'(x)dx = f(b) - f(a)$ can be perceived as a special case of the path independence theorem in the case of line integral.
  According to Niss (1999) one of the major findings of research in mathematics education is the key role of domain specificity. The student's conception of a mathematical concept is determined by the set of specific domains in which that concept has been introduced for the student. By expanding the domain the problem of domain specificity would be transcended.
- On the other hand, Taylor's theorem or Taylor approximation may not be thought of as a special case of Fourier Theorem. But the similarity is clear in that both of

them approximate particular types of functions into either a power series (in the case of Taylor's theorem) or a trigonometric series (in the case of Fourier theorem), with adequate conditions in both cases. Also in both cases approximations are made for the purpose of simplifying a function. The "transition" from Taylor series approximation to Fourier series approximation can be considered as a (stretched) abstraction, where specific properties have been abstracted and applied to a different context for similar purposes.
Recall that generalization is the process of forming general conclusions from particular instances while abstraction is the isolation of specific attributes of a concept so that they can be considered separately from the other attributes. Yet, abstraction is often coupled with generalization, but the two are by no means synonymous. Any arguments which apply to the abstracted properties apply to other instances where the abstracted properties hold, so (provided that there are other instances) the arguments are more general (Tall, 1988).

In lieu of a proof?

As instructors one should ask: is a mere display of the natural transition a substitute for a thorough proof or justification along the lines of a "scientific proof"? To what degree shall we be satisfied with "observing" a similarity in the results and just accept the extensions in all those listed illustrations as new natural facts? In the case of engineering students whose programs do not require knowledge of thorough justification in the form of formal proof, is observing the smoothness of the extensions between cases of functions of one variable and functions of several variables considered as a good enough justification of the new result as a substitute for what is referred to as "scientific proof". Is it enough to draw students' attention to the fact that Green's theorem is a generalization of the Fundamental Theorem of Calculus, and Stokes theorem is a generalization of Green's Theorem for different dimensions so to speak? No need to mention that some students are able to see the connection between those results with no help from the instructor. Once the generalized result is presented to them, they look back at what could have been its source and figure out in retrospect how it could have been anticipated. In all cases, isn't it an innate instinctive drive to want to connect and look at things as part of one big whole entity or behavior?

General questions:

1. To what extent does drawing the students' attention to the transition in a natural way a substitute for a formal thorough proof that highlights conditions and particularities of the two domains?
2. How much emphasis should be placed on the conditions and the particularities of both domains without menacing mathematical intuition and free "guessing" skills?
3. What are the cognitive skills involved in the generalizations and or abstraction? And are the students ready for that level of cognition at that stage?
4. Should transition be solicited or just lightly pointed out to (in retrospect)?

5. What instructional procedures should be followed that guide students to come to terms with the second type of transition on their own?

Conclusion:

One disadvantage of this behavior would be creating overconfident students. For instance, in probability a student might get tempted to expand the case of binary outcomes of a coin to the case of a die. As for most of us, we must have learned such transitions mostly in retrospect, since the traditional textbooks we used do not bother pointing to those connectives: once we saw the generalized result, we must have figured out its source of inspiration hoping that next time around, we can anticipate it ourselves. One positive outcome is bound to happen, and that is seeing mathematics as a collection of connected facts.

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