Reflecting Problem Orientation in Mathematics Education within Teacher Education

Günter Graumann, Prof. Dr.
Retired Professor of Mathematics and Didactic of Mathematics
Faculty of Mathematics, University of Bielefeld
Universitätsstrasse 25, D-33615 Bielefeld, Germany
graumann@mathematik.uni-bielefeld.de

Summary: Problem orientation is an important aspect within mathematics education we all know. But we also know that problem orientated mathematics teaching is practices in school reality rarely. To get a change the paradigm of mathematics teaching need a transformation.
I think besides theoretical discussions within the didactical community and the presentation of interesting proposals for different classroom conditions - which are important - first of all the teachers must be familiar with and get a positive belief about problem orientation in mathematics education.
In my presentation I will report about experiences in respect to problem orientation I did make with teacher students at our university in Bielefeld.

Problem orientation is an important aspect within mathematics education we all know. It is demanded since more then fifty years. The students first can learn a lot of mathematics on this way. Secondly they learn mathematics as a process. But most important is that they can train several general goals as

- how to handle a problem, how to make investigations,
- ability of problem solving and finding analogies
- why and how to order and systematize a given situation
- positive belief about mathematics and self-confidence .

But in most countries its realisation in school leaves a great deal to be desired. Therefore it is important to support a teaching which contains a lot of acting with problems.
For this - besides the necessary theoretical discussions within the didactical community and the presentation of interesting proposals for different classroom conditions - ***first of all the teachers must be familiar with and get a positive belief about problem orientation in mathematics education***. The teachers must have own experiences with working on single problems as well as problem fields. Moreover they must know about and reflect upon fundamental ideas about heuristics, problem solving and problem finding or variation of problems. Moreover they should be able to find different ways of working with a problem and be open for new approaches used by the students.
Here I will report about experiences in respect to problem orientation I did make with teacher students at our university in Bielefeld.

Survey of four seminars

Besides integrating some theoretical and practical aspects about problem orientation into all my lectures and seminars in the last time I offered four special seminars concentrating on problem orientation in mathematics education. Three of these seminars have been seminars for preparing teacher students for writing a final Bachelor paper. For all of these about 20 students per seminar I determined that the theme of each paper should refer to problem orientation. The other seminar was a normal one for senior teacher students within their Bachelor study. In all of these three seminars on one hand the students had to work by themselves with different problems and find new problems within the discussed problem field – I here already will mention that the last task was very hard for the students – and on the other hand I made some inputs with copies out of literature. Of course they also had to reflect on mathematics teaching with special concern to problem orientation.

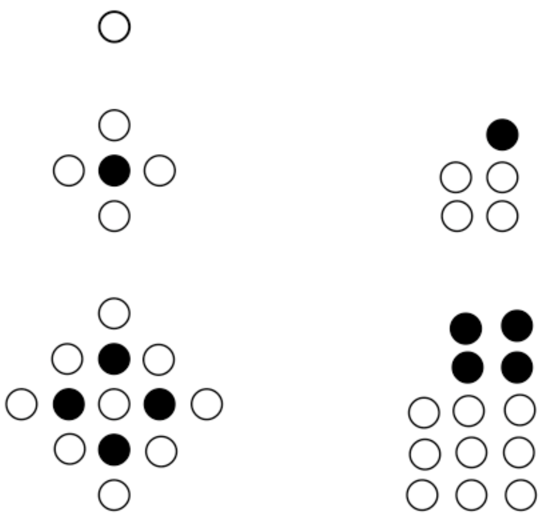
Before going into details I would like to give an overlook of the topics we discussed. In all of the seminars in the first session (lasting one and a half hour) I presented *three different problems to work on by themselves* (together with their neighbours) without giving any help or hint. I will report on the results of this session later on.
Lateron we discussed with help of literature and internet the following themes: *History of problem orientation as well as learning by discovery, learning by doing and self regulation, definitions of "What is a problem", methological hints for problem solving, heuristics and problem orientation* and also *types of tasks, types of problems and ways of developing tasks by your own, self-activity and self-regulation in the discussion of didactics of mathematics within the last ten years* (cf. e.g. Pehkonen & Graumann 2007, Büchter & Leuders 2005 and Polya 1949).
We also deepened some theoretical aspects like *variation of taks* (according to Schupp 2004), "*logic of failure*" (according to Dörner 1989), *mathematical learning from constructive view*, *beliefs about problem orientation*, *barriers in respect to changing mathematics teaching*, *aims of and motivation for problem orientation*, *statements according to problem orientation in official guidelines*.
Mathematical topics considering the aspect of problem orientation have been e.g.:
- *Figured numbers and special sums, Partition of sets and representations of numbers as sums in grade 1, sequences and chains of numbers, number trains, number walls* (cf. Graumann 2009)
- *Magic squares and sudoku - a topic for grade 2 and 4, Polyominos – a geometrical problem field for grade 3, distribution of prime numbers in grade 3, division with rest in grade 4*
- *Triangles with integers as side length, regular polygons and polygons in space, Pythagorean triples,*
- *problems from PISA, Fermi tasks*

The "Mason problem" as inspirer
Some years ago I did hear from a seminar in Debrecen where John Mason as guest was present. In this seminar John Mason asked the participants (mathematics teacher students and secondary mathematics teacher) to solve the following problem.

$1 =$

$1 + 3 + 1 =$

$1 + 3 + 5 + 3 + 1 =$

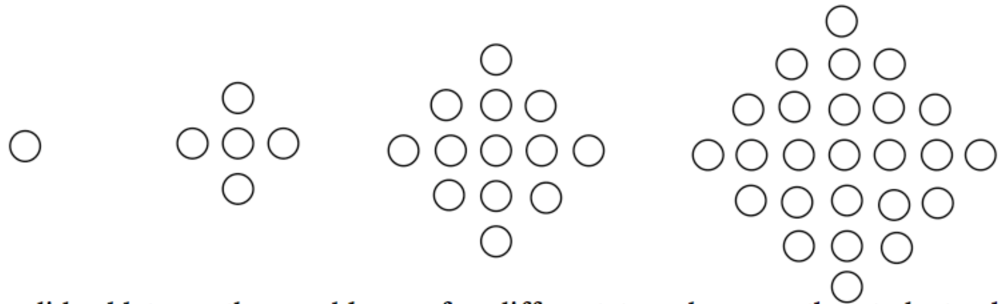


Task: "*Formulate a general question for this problem! Try to formulate a conjecture to your question! Prove the conjecture!*"

After 10 to 15 minutes it was clear that such open problems are very uncommon to Hungarian students and teacher, most of them could not do anything. This caused me to present the "Mason problem" to my students at the beginning of the seminar so that they are not influenced by discussions within the seminar (even though the title of the seminar may have influence and in their first semester they have already seen simple figured numbers like square numbers and triangle numbers). I wanted to know how they will act on this problem.

I varied the presentation of the Mason problem a little bit in that way not giving symbolic hints in respect to sums of numbers and not painting the little circles different because I wanted to see how the students will do it by themselves and whether they will discover different structures.

Two rows of the figures shown below have been given with the following text: "*Draw the next two figures of this sequence. Look out for partial figures and mark them. Which arithmetical representation do you can find on this way?*"



I also did add two other problems of a different type because the students should make experiences with different types of problems too. These two other problems have been given in text in the following way.

Problem concerning a ghost of a river: *A ghost of a river says to a walker who just will cross a bridge: "If you cross the bridge I will double the money you have in your pocket; but if you go back across the bridge I will take 8 Euros away from your pocket." When the walker came back the third time the money in his pocket was gone away (exactly 0 Euros).*

Problem concerning small animals: *Grandfather Miller has in his yard hens and rabbits. Once upon a time he counted 7 heads and 20 legs. (Variation: He did count only 20 legs).*

Different results of these two problems given in text

First of all I can tell that all of the students except one in the discussion at the end of the first double hour reported that an open problem like that from Mason was very new for them. But all of them (at least together with a neighbour) started to work on that problem and most of them got at least one special result. And it could be noticed that in any of the seminars There appeared different ways of working with these three problems.

In the following I will present all different ways of working with these problems; in doing so I will start with the two problems given in text.

Problem concerning a ghost of a river

1. Method of trial and error with variation: We start with 6. The transformation from this are $6 \to 12 \to 4 \to 8 \to 0 \to 0 \to$ not possible. We see that we have to get 6 after the first crossing and the way back. Thus we try it with two more Euros and get $8 \to 16 \to 8$

$\rightarrow 16 \rightarrow 8 \rightarrow 16$ and find an endless sequence. Now we try the number between and get the solution $7 \rightarrow 14 \rightarrow 6 \rightarrow 12 \rightarrow 4 \rightarrow 8 \rightarrow 0$.

(By varying our thoughts we could start with 5 or 4 and see that all numbers of the sequence decrease just like starting with 9, 10, … will let increase all numbers of the sequence. That means we may find a functional relation.)

2. Method of working backwards: $0 \leftarrow 8 \leftarrow 4 \leftarrow 12 \leftarrow 6 \leftarrow 14 \leftarrow 7$.
3. Method with algebraic formula: $2 \cdot (2 \cdot (2 \cdot x - 8) - 8) - 8 = 0$ or

$x \rightarrow 2x \rightarrow 2x\text{-}8 \rightarrow 2\cdot(2x\text{-}8) \rightarrow 2\cdot(2x\text{-}8)\text{-}8 \rightarrow 2\cdot(2\cdot(2x\text{-}8)\text{-}8) \rightarrow 2\cdot(2\cdot(2x\text{-}8)\text{-}8)\text{-}8$

From $2\cdot(2\cdot(2x\text{-}8)\text{-}8)\text{-}8 = 0$ we will get $x = 7$.

We can get a generalisation via this method with $2 \rightarrow a$, $8 \rightarrow b$ and $3 \rightarrow n$:

$x \rightarrow ax \rightarrow ax\text{-}b \rightarrow a\cdot(ax\text{-}b) \rightarrow a\cdot(ax\text{-}b)\text{-}b \rightarrow a\cdot(a\cdot(ax\text{-}b)\text{-}b) \rightarrow a\cdot(a\cdot(ax\text{-}b)\text{-}b)\text{-}b$

$\rightarrow a\cdot(a\cdot(a\cdot(ax\text{-}b)\text{-}b)\text{-}b) \rightarrow a\cdot(a\cdot(a\cdot(ax\text{-}b)\text{-}b)\text{-}b)\text{-}b \ldots \rightarrow a^{n\cdot}x - (a^{n-1} + a^{n-2} + \ldots + 1)\cdot b$

Problem concerning small animals

1. Method of trial and error with variation: We try 4 rabbits → 16 feet, with the left 4 feet we get 2 hens; that make together 6 heads. Because one head is undercharged we have to increase the number of hens respectively decrease the number of rabbits. 3 rabbits → 12 feet and 4 hens → 8 feet gives 7 heads and 20 feet as desired.
2. Method of working backward from the heads: Any animal has at least 2 feet, so with 7 heads we have at least 14 feet. The rest of 6 feet is going in pairs to 3 rabbits, so we get 3 rabbits and 4 hens.
3. Method with algebraic formula: x = number of rabbits, y = number of hens. $4x + 2y = 20$ (number of feet) and $x + y = 7$ (number of heads) . Then solving with algebraic instruments gives $x = 3$, $y = 4$.

Variation of this Problem

The variation shall show the students that we also can get a problem that has more than one solution and we have to find a systematic for finding all solutions.

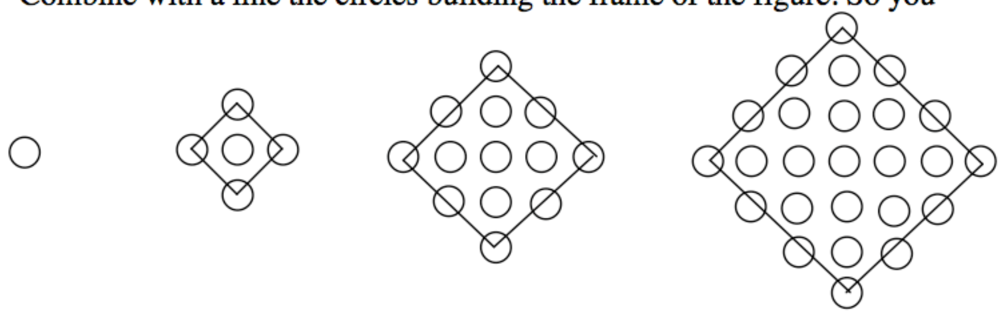
Here the minimal number of heads is 5 because 5 rabbits and 0 hens makes 20 feet. Reducing the number of rabbits step by step you will get 6 heads (4 rabbits and 2 hens), 7 heads (3 rabbits and 4 hens), 8 heads (2 rabbits and 6 hens), 9 heads (1 rabbit and 8 hens), 10 heads (0 rabbits and 10 hens). The solution with 0 hen and that one with 0 rabbit probably does not fit to the text and thus these solutions have to be erased.

In addition we can make investigations in respect to functional relations like "Reducing the number of rabbits by one causes increasing the number of hens with two" or "Reducing the number of heads by one causes decreasing the number of hens with two".

Different ways the students worked with the "Mason Problem"

The following different groupings by colouring some circles or combining some circles with a line and symbolic descriptions have been the following:

1. Combine with a line the circles building the frame of the figure. So you

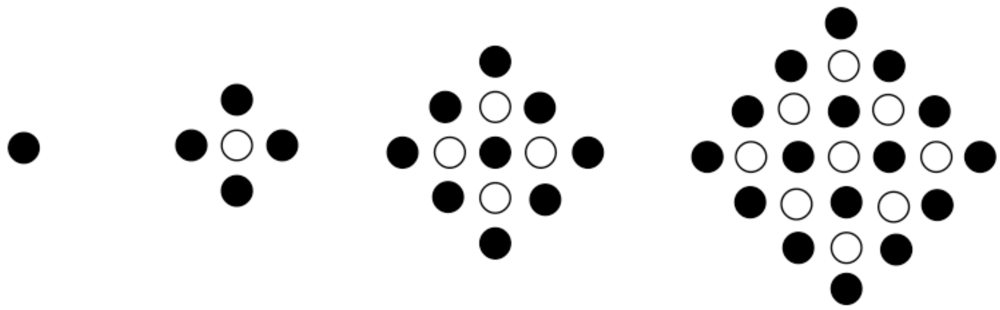


get the description 1, 1+4, 1+4+8, 1+4+8+12, … and in general $1 + 4 \cdot [1+2+3+\ldots+(n-1)]$.

If we already know that $1+2+3+\ldots+(n-1) = \frac{1}{2} \cdot (n-1)\cdot n$ we will get the general symbolic description $1 + 2\cdot(n^2-n)$ [resp. $2n^2-2n+1$].

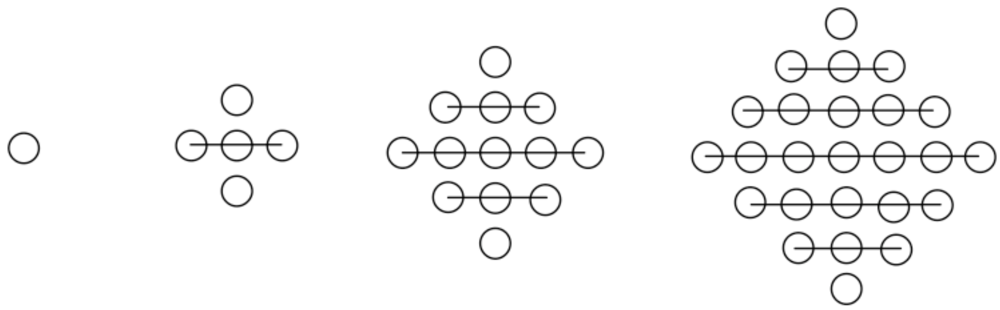
2. In a more arithmetical view on these figures some students looked at the respective number of circles: 1, 5, 13, 25, … . From this they detected that the difference sequence is built by the multiple of 4. On this way they got the same general description as above.

3. Some students coloured the circles in the frame together with inner circles for getting a squared number. The non-coloured circles then built a squared number too but a smaller one, more precisely the length of the side is one less than the length of the side of the coloured square.



The symbolic description thus came out as 1^2, 2^2+1^2, 3^2+2^2, 4^2+3^2, … or in general $(n-1)^2 + n^2$ [respectively $2n^2 - 2n + 1$].

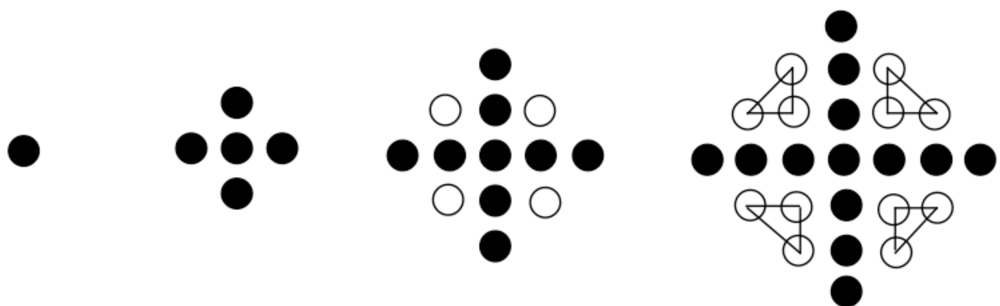
4. A fourth group of students looked at the horizontal (or vertical) rows and got the symbolic description 1, 1+3+1, 1+3+5+3+1, 1+3+5+7+5+3+1, … .



If we already know that the numbers 1, 1+3, 1+3+5, 1+3+5+7, … describe a square number we can see the identicalness with the symbolic descriptions above.

5. One student coloured the vertical and horizontal middle lines building a cross. The non-coloured circles then build four triangle figures and we get 1, $1 + 4\cdot1$, $1+ 4\cdot2 + 4\cdot1$, $1 + 4\cdot3 + 4\cdot[1+2]$, … .

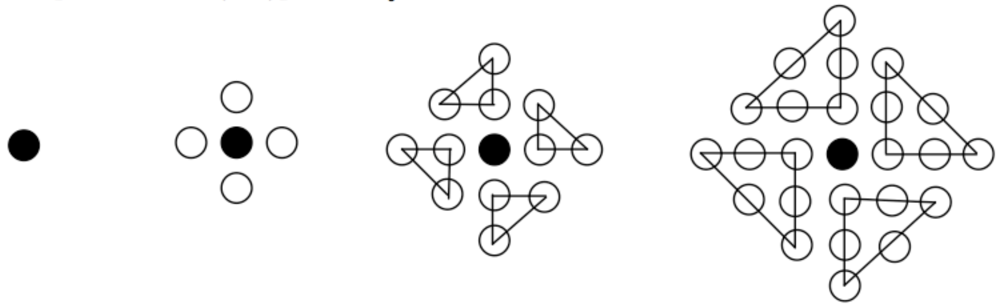
6.



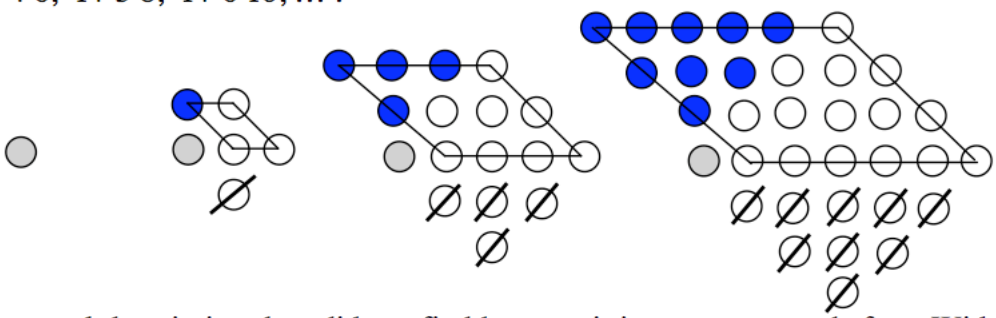
If we want to get a symbolic description in general then we can find

$1+4\cdot(n-1)+4\cdot[1+2+3+\ldots+(n-2)]$ resp. $1+4\cdot[1+2+3+\ldots+(n-1)]$.

7. Two students also saw four triangle numbers but including always one branch of the cross so that only the middle point is extra standing. This leads to the formula $1+4\cdot[1+2+3+\ldots+(n-1)]$ directly.



8. Another group of students resorted the figures of circles by erasing the rows below the horizontal middle line and then added these circles on the left side of the remaining circles so that after that all horizontal line have the same length without the bottom line which has one circle more. So they got the sequence 1, $1+2\cdot2$, $1+3\cdot4$, $1+4\cdot6$, $1+5\cdot8$, $1+6\cdot10$, … .



A general description they did not find because it is no so easy as before. With some considerations you can find the formula $1+n\cdot(2\cdot(n-1))$ resp. $1+2n^2-2n$.

In the discussion with the whole group the different solutions were presented with adding the missing symbolic descriptions. For teacher students this is very important because they can see the large variety of working on such a problem. Later on as teacher in school on one hand it is important to be open for different ideas of pupils and on the other hand the arrangement for working with problems should include working in small groups as well as reflecting different approaches and their connections.

References

Büchter, A. & Leuders, T. (2005). Mathematikaufgaben selbst entwickeln, Cornelsen: Berlin

Dörner, D. (1989). Die Logik des Misslingens – Strategisches Denken in komplexen Situationen, Rowohlt: Reinbek.

Graumann, G. (2009). Problem Orientation in Primary School. In:

Graumann, G. & Pehkonen, E. (2007). Problemorientierung im Mathematikunterricht – ein Gesichtspunkt der Qualitätssteigerung. In: Teaching Mathematics and Computer Science, University of Debrecen, 5/1 (2007), 251-291.

Leuders, T. (2003). Mathematik-Didaktik – Praxishandbuch für die Sekundarstufe I und II, Cornelsen: Berlin

Polya, G. (1949). Schule des Denkens – Vom Lösen mathematischer Probleme, Francke: Bern

Polya, G. (1964). Die Heuristik – Versuch einer vernünftigen Zielsetzung. In: Der Mathematikunterricht, Heft 1/1964, 5 – 15.

Schupp, H. (2002). Thema mit Variationen oder Aufgabenvariation im Mathematikunterricht, Franzbecker: Hildesheim