School-Mathematics all over the World – some Differences

Ludwig Paditz, University of Applied Sciences Dresden, Germany
paditz@informatik.htw-dresden.de

Abstract:
The lecture is devoted on some differences of definitions in school-mathematics. In other countries we have sometimes other definitions compared with Germany. Sometimes we can discover even in Germany some differences, because we have in Germany 16 smaller federal countries and every country has its own ministry of education.
On the other hand every professor in any university is free and independent concerning the research and education. He could use his own definition in special fields of mathematics. We would have no problems, if we respect different definitions of different teachers and in different software:

1. What is the set of natural numbers?
   $\{0, 1, 2, \ldots\}$ or $\{1, 2, 3, \ldots\}$ and what means the symbol **N**?
2. What means $3^1/_2$? Is it $3 + {}^1/_2 = 3.5$ or $3 * {}^1/_2 = 1.5$?
3. Where the real function $y = \mathrm{f}(x) = x$^(1/3) is defined? x must be non-negative or $x \in \mathbf{R}$?
4. What means the notation $y = \tan^{-1}(x)$? Is this the arctan-function or the cot-function?
5. What is $y = \log(x)$? We know $y = \log_a(x)$. What is the base a in $y = \log(x)$?
   Without any a we know $y = \lg(x)$ or $y = \ln(x)$ or $y = \mathrm{lb}(x) = \mathrm{ld}(x)$ (older notation).
   Sometimes we find the notation $y = {}_a\log(x)$. Why different notations?
6. What is the main argument φ of a complex number in the Gaussian plane?
   $\varphi = \arg(-2\text{-}2\mathbf{j}) = -3\pi/4$ or $\varphi = \arg(-2\text{-}2\mathbf{j}) = 5\pi/4$?
7. What is the 3rd main root of the complex number -2-2**j** ? What is (-2-2**j**)^(1/3) ?
   What is (-8)^(1/3) ? Draw the real function $y = \mathrm{f}(x) = x$^(1/3) $\approx x$^0.33333333 !
8. What is the α-quantile x_α of a probability distribution of a random variable X?
   x_α is a number with $\mathrm{P}(\mathrm{X} < x_\alpha) \le \alpha \le \mathrm{P}(\mathrm{X} \le x_\alpha)$ or $\mathrm{P}(\mathrm{X} < x_\alpha) \le 1\text{-}\alpha \le \mathrm{P}(\mathrm{X} \le x_\alpha)$
9. Is the distribution function $y = \mathrm{F}(x)$ of a random variable X right continuous or left continuous? Is $\mathrm{F}(x) = \mathrm{P}(\mathrm{X} < x)$ or $\mathrm{F}(x) = \mathrm{P}(\mathrm{X} \le x)$? Consider the binomial distribution!

We have to solve in this context two basic problems:
A. If students change the school or university (this is the mobility of our young people) and go to another country, than they remark or not remark that some differences exist in mathematical definitions. In the examination sometimes we observe some mistakes of our students and the reasons are some differences in education.
B. We observe sometimes differences between teaching and used software in calculators or PC or used books of several authors.

In the lecture we will discuss about the stated problems and show some examples by the help of CASIO **ClassPad330** (operating system 3.06, published 2011).
The well known software package **Mathematica** (Version 8) by Wolfram could be the basic for all teachers in the world to work with standard definitions which are used in Mathematica.
In Germany the "**German Institute for Standardization**" (Deutsches Institut für Normung, **DIN**) offers stakeholders a platform for the development of standards as a service to industry, the state and society as a whole. DIN has been based in Berlin since 1917. DIN's primary task is to work closely with its stakeholders to develop consensus-based standards that meet market requirements. By agreement with the German Federal Government, DIN is the acknowledged national standards body that represents German interests in European and international standards organizations. Ninety percent of the standards work now carried out

by DIN are international. Standards play a major deregulatory role. DIN's goal is to develop standards that have validity worldwide.
ISO (the **International Organization for Standardization**) is the world's **largest developer** and publisher of **International Standards**. It has its headquarters in Geneva, Switzerland. ISO considers this trend of utmost importance and believes in the fundamental contribution that educational institutions can give on teaching what international standardization is and what can be achieved through it. Cp. chapter 10, **SANS** (South African national standard).

References:

http://en.wikipedia.org/wiki/International_Organization_for_Standardization
http://www.din.de/cmd?level=tpl-home&languageid=en
http://www.iso.org/iso/home.htm
http://edu.casio.com/products/classpad/
http://www.wolfram.com/

1. What is the set of natural numbers?

There are two conventions for the set of natural numbers: it is either the set of positive integers {1, 2, 3, ...} according to the traditional definition; or the set of non-negative integers {0, 1, 2, 3, ...} according to a definition first appearing in the 19th century.

Have a look in **DIN5374**(logic and set theory, symbols and concepts) or **DIN1302**(general mathematical symbols and concepts) or **ISO31-11**(quantities and units – part 11: Mathematical signs and symbols for the use in physical sciences and technology, revised by **ISO 80000-2**:2009): $\mathbf{N}=\{0, 1, 2, \ldots\}$ and $\mathbf{N}^*=\{1, 2, 3, \ldots\}$,
see: http://en.wikipedia.org/wiki/ISO_31-11

e.g. in the online-book
MATHEMATICAL PREPARATION COURSE BEFORE STUDYING PHYSICS
http://www.thphys.uni-heidelberg.de/~hefft/vk_download/vk1e.pdf (March 10, 2011)
you can read:
"We begin with the set of natural numbers {1, 2, 3, …}, given the name **N** by mathematicians and called "natural" because they have been used by mankind to count within living memory." Here $\mathbf{N}_0 = \{0, 1, 2, 3, \ldots\}$ (in Germany: ISBN 978-3-8274-1638-4)

2. What means $3^1/_2$?


http://lamar.colostate.edu/~hillger/faq-images/faq-frac.jpg

Using metric measurements, you don't need to do arithmetic with mixed numbers.

What means $3^1/_2$? Often our students are not sure is it $3 + {}^1/_2 = 3.5$ or $3 * {}^1/_2 = 1.5$?
For our students a problem of the mixed notation is that it can be misinterpreted as a product.
A **mixed number** is the sum of a whole number and a proper fraction.

In the expression $\mathbf{a}\ {}^{\mathbf{b}}/_{\mathbf{c}}$ is omitted the operator. Here it is a multiplication, because in terms with symbolic variables other arithmetic operators can not be omitted, cp. DIN1302.

The multiplication of quantity symbols (or numbers in parentheses or values of quantities in parentheses) may be indicated in one of the following ways: ab, a b, a·b, a×b.
When the dot is used as the decimal marker as in the United States, the preferred sign for the multiplication of numbers or values of quantities is a cross (that is, multiplication sign) (×), not a half-high (that is, centered) dot (·).
Example: Write 15×72 but not 15·72 , cp. http://physics.nist.gov/cuu/pdf/sp811.pdf

In some countries such as France, the mixed number is unusual.
See: http://en.wikipedia.org/wiki/Fractions and http://de.wikipedia.org/wiki/Bruchrechnung

You may, of course, ***say*** "three and a half" — 0.5 is often read aloud as "one half" — but you should always ***write*** it as a decimal fraction.
http://lamar.colostate.edu/~hillger/faq.html
http://lamar.colostate.edu/~hillger/decimal.htm

Why Decimal?

A room measures **15**ft. **3-3/4**in. by **21**ft. **7-1/2**in. (**4.667**m by **6.591**m).
Questions:

What is its floor area in *square yards*?
What is its floor area in *square meters*?

Answers:

36.79sq.yd., or **30.76**m^2

3. Where the real function *y* = f(*x*) = *x*^(1/3) is defined?

The 3rd root is a special case of exponentiation (real power with a fraction), cp.
http://en.wikipedia.org/wiki/Exponentiation

We know:
The exponentiation operation with integer exponents requires only elementary algebra.
By definition, raising a nonzero number to the −1 power produces its reciprocal.
Raising a positive real number *x* to a power that is not an integer, say 1/3, can be accomplished in two ways.

a) Rational number exponents can be defined in terms of n^{th} roots, and arbitrary nonzero exponents can then be defined by continuity: *x*^(1/3) ≈ *x*^0.333333333 = (*x*^333333333)^(1/1000000000) = (*x* ^(1/1000000000))^333333333 (the exponent is only a limit of a sequence with finite decimal numbers). Thus it seems, *x* can't be negative.
b) The natural logarithm can be used to define real exponents using the exponential function: *x*^(1/3) = exp(ln(*x*)/3). Thus it is clear, *x* can't be negative (and not zero)

In DIN1302: *y* = *x*^(1/n) is the n^{th} root **with a positive *y*** such that *y*^n=*x*. Here n∈**N*** and *x* is a nonnegative real number. In many **German school books** we find this definition of the n^{th} root in the domain of the real numbers.
e.g. Definition 1.6 in ISBN 978-3-427-21503-5(2007: Mathematik für Berufliche Gymnasien)

Another question is the real solution of the equation x^3 = -8.
The real solution is -8^(1/3)= -(8^(1/3)) = -2 but not (-8)^(1/3) or (-8)^0.333333333.
The last numbers are not real but complex (cp. principal value, complex main root in **C**).

In US-school books a negative base is allowed:
For any real numbers *a* and *b*, and any positive integer *n*, if $a^n = b$, then *a* is the n^{th} root of *b*.
Here (-8)^(1/3) = -2, where from the point of view of **C** the value -2 is not the principal root.

http://www.farmersville.k12.ca.us/aztecs/Department/Math/spradling/Algebra%20II/Alg%202%20unit%207/Review%20&%20tests/Ch%20review%20answers.pdf

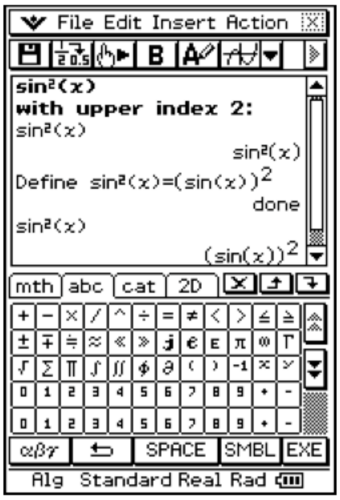
ISBN 0-618-39478-8 (2005: Precalculus with Limits – A graphical Approach)
ISBN 0-471-48273-0 (2005: Calculus)

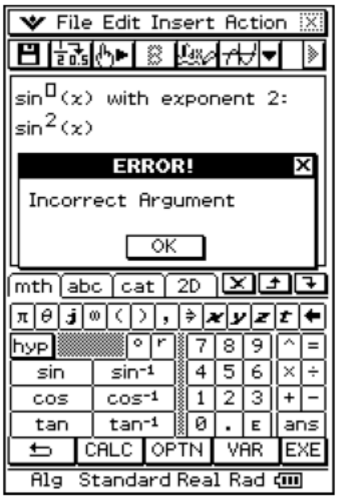
4. What means the notation $y = \tan^{-1}(x)$?

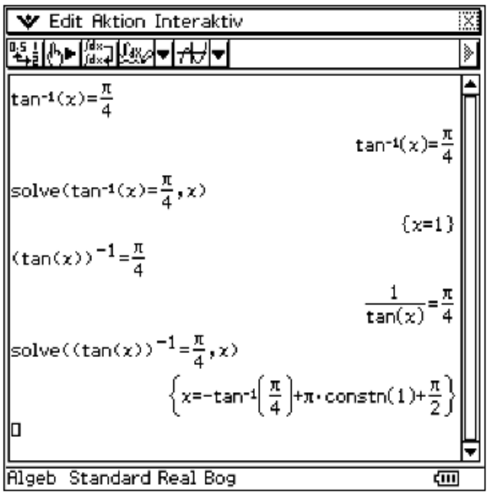
Is this the arctan-function or the cot-function?
For a given function y = f(x) we have the notation y = $f^{-1}(x)$ for the **inverse function**.
However the notation $(f(x))^{-1}$ means 1/f(x). Here we have the **exponentiation with -1**.

In textbooks we often read e.g. $\sin^2(x)+\cos^2(x) = 1$ (trigonometric Pythagoras)
http://www.qc.edu.hk/math/Certificate%20Level/Trigo%20Py%20Th.htm

If you work with a calculator: the input $\sin^2(x)+\cos^2(x) = 1$ is not defined. You have to write $(\sin(x))^2+(\cos(x))^2 = 1$ or you have to define the symbol $\sin^2$ and $\cos^2$ (a name with four characters!). On the other hand the symbol $\tan^{-1}$ is clear for the calculator.







Only in the case if the symbolic exponent equals -1 our students have problems e.g. with $y = \tan^{-1}(x)$. The calculators use the notation $\tan^{-1}(x)$, but in written form the students should use the notation arctan(x) (DIN1302) and not $\tan^{-1}(x)$ to avoid problems and misunderstandings.

5. What is $y = \log(x)$?

We know $y = \log_a(x)$. What is the base a in $y = \log(x)$?
Without any a we know $y = \lg(x)$ or $y = \ln(x)$ or $y = \text{lb}(x) = \text{ld}(x)$ (older notation). Sometimes we find the notation $y = {}_a\log(x)$. Why different notations? We should use modern notations given by DIN and ISO. We should avoid the old notations ld(x) and ${}_a\log(x)$. In calculators log(x) means lg(x). Why not a lg-key instead of log-key? In ISBN 978-3-8274-1638-4 (and English and Spanish translation 2011) again the old notation, e.g. ld(x) = ${}_2\log(x)$ – why ?

6. What is the main argument φ of a complex number in the Gaussian plane?

φ = arg(-2-2**j**) = -3π/4 or φ = arg(-2-2**j**) = 5π/4 ?

In every calculators and PC-software and DIN1302 the main argument is in the interval]-π, π] = (-π, π], but in many school books we can read: the main argument is in the interval [0, 2π[= [0, 2π).

Mathematica: In[1]:= Sqrt[-1] Out[1] = **i** In[2]:= Arg[-2-2**i**] Out[2] = -3π/4

ClassPad: √ (-1) = **j** arg(-2-2**j**) = -3π/4

We have two notations for the complex unit: **i** in mathematics and **j** in engineering.

In the engl. translation (2011) of the book ISBN 978-3-8274-1638-4 you can read:

"for the (only modulo 2π determined) argument of a complex number:
$0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi$"

Here $z = x + y\mathbf{j}$, i.e. (x, y) are the Cartesian coordinates of z. Why here not include $\varphi = 0$?

We know the real function f(t) = arctan(t) has its values in the interval]- π/2, π/2[= (- π/2, π/2), thus the definition **$0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi$** is not correct!

In the German online script http://www.thphys.uni-heidelberg.de/~hefft/vk_download/vk1.pdf (March 11th, 2011) the author has added a remark:

„**Einschub: Alternative Phasenkonvention:** Natürlich kann man auch ein um den Ursprung symmetrisches Intervall der Länge 2π für die Argumente der komplexen Zahlen wählen: **$-\pi < \varphi = \arg z := \arctan(y/x) \leq \pi$**, was allerdings später die Herleitung der Wurzelfunktionen etwas komplizierter macht. Nach Prof. L. Paditz von der HTW Dresden empfiehlt die DIN diese Konvention."

Here the author is again not correct: He has to write **$-\pi < \varphi = \arg z \leq \pi$** whithout the arctan-function! Furthermore he remarks that with the DIN convention for the argument φ the definition of the root-function will be more difficult – this is not true! The definition of the root based on the main argument **$\varphi = \arg z$** is very easy:

The main root is **z_0 = z^(1/n) = |z|^(1/n) * exp(arg(z)j/n)**
and the other roots are **z_k = z_0 * exp(k*2πj/n), k = 1, 2, …, n-1.**

This definition of the main root and the other roots is implemented in all calculators and PC-software.

ClassPad: arg(x+yi)|x<0 yields the correct main argument **$\tan^{-1}$(y/x)+signum(y)*π**

Spanish version (March 11th, 2011) of the considered online book „MATHEMATICAL PREPARATION COURSE BEFORE STUDYING PHYSICS":

CURSO DE MATEMÁTICA PREPARATORIO - para el estudio de la Física
traducido por Prof. Dr. LAUTARO VERGARA, Universidad de Santiago de Chile

http://www.thphys.uni-heidelberg.de/~hefft/vk1/k0/000s.htm

"Argumento el número complejo: **$0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi$** (sólo determinado módulo 2π)"

http://www.thphys.uni-heidelberg.de/~hefft/vk1/k8/814s.htm

Thus an obvious error goes around the world!

7. What is the 3rd main root of the complex number -2-2j ?

What is (-2-2j)^(1/3) ?

What is (-8)^(1/3) ? Draw the real function $y = f(x) = x$^(1/3) ≈ x^0.33333333 !

The main root is z_0 **= z^(1/n) = |z|^(1/n) * exp(arg(z)j/n) = 1 – j** if z =-2-2**j**

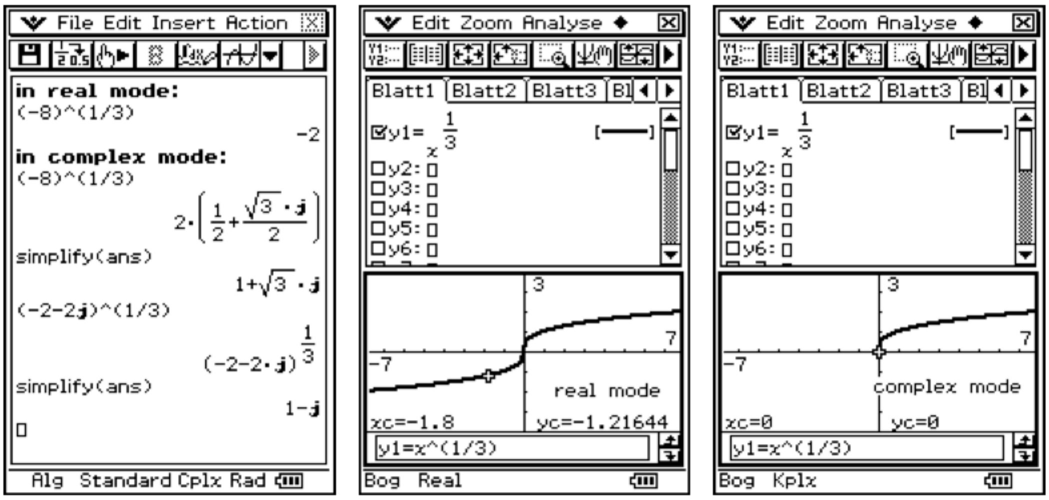
Here we use the main argument **arg(z)** = -3π/4, i.e. **arg(z)j/n** = -π/4.

If we follow the other point of view with **arg(z)** = 5π/4, i.e. **arg(z)j/n** = 5π/12, we would get the main root (-2-2**j**)^(1/3) = **√(3)/2 – 1/2 + (√(3)/2 + 1/2) j** and not **1-j.**

The same with (-8)^(1/3). The main root is 2*exp(πj/3) = **1 + √(3)j** and not **-2.**

ClassPad: in complex mode: (-8)^(1/3) = **1 + √(3)j, in real mode:** (-8)^(1/3) = -2.

In the real mode the ClassPad follows the US-definition of a root, thus we get here different graphics of $y = f(x) = x$^(1/3) in the x-y-plane, in dependence on real mode or complex mode:



However in real mode or complex mode we get the same graphics for y =f(x)= x^0.33333333!

8. What is the α-quantile x_α of a probability distribution of a random variable X?

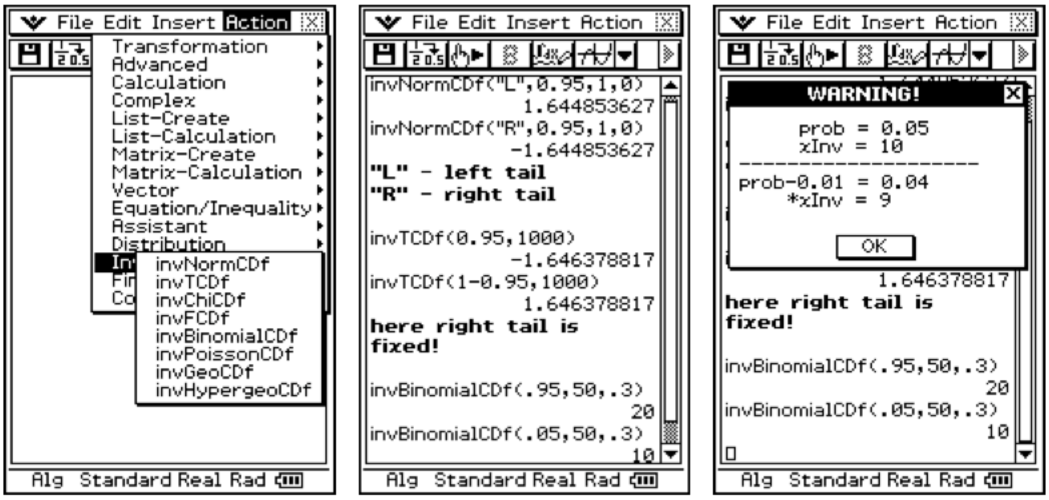
x_α is a number with $P(X< x_\alpha) \le \alpha \le P(X \le x_\alpha)$ or $P(X< x_\alpha) \le 1-\alpha \le P(X \le x_\alpha)$?

Follow **Mathematica "Quantile[dist, α] is equivalent to InverseCDF[dist, α]"** we get the definition: $P(X< x_\alpha) \le \alpha \le P(X \le x_\alpha)$, i.e. according to DIN ISO 3534-1 (2009) (Statistics – Vocabulary and symbols – Part 1: General statistical terms and terms used in probability) based on ISO 3534-1 (2006) (http://it.wikipedia.org/wiki/ISO_3534).

For a continuous distribution of X the inverse CDF at α is the value x_α such that **CDF[dist, x_α]= α.** For a discrete distribution of X the inverse CDF at α is the smallest integer x_α such that **CDF[dist, x_α] ≥ α.**

cp.
http://reference.wolfram.com/mathematica/ref/Quantile.html
http://reference.wolfram.com/mathematica/ref/InverseCDF.html
http://reference.wolfram.com/mathematica/ref/CDF.html
http://reference.wolfram.com/mathematica/ref/Probability.html

However ClassPad only follows this definition for discrete distributions but for continuous distributions ClassPad works with the other definition **P(X< x_α) ≤ 1-α ≤ P(X ≤ x_α)**.



The last picture shows a good service during the computing the quantile of a discrete distribution. If the assumed probability (say $\alpha = 0.05$) is no value of the distribution function, than the computed quantile is the smallest integer x_α such that $\alpha \le$ P(X ≤ x_α) = F(x_α), i.e. for x_α-1 we have already F(x_α-1) = P(X ≤ x_α-1) = P(X< x_α) < α . The "WARNING!" is connected with the last digit after the decimal point of the assumed probability prob = α and will not appear, if a change in the last digit by one step will not change the computed quantile.

9. Is the distribution function *y* = F(*x*) of a random variable X right continuous or left continuous?

Is F(x) = P(X< x) or F(x) = P(X≤ x) ? Consider the binomial distribution!

For a continuous distribution both definitions of F(x) are equal, but for discrete distributions we get here a left continuous function with F(x) = P(X< x) and a right continuous function with F(x) = P(X≤ x). The last definition is in context with **DIN ISO 3534-1 (2009)** based on **ISO 3534-1 (2006)** and all software, e.g. Mathematica or ClassPad.

http://en.wikipedia.org/wiki/Cumulative_distribution_function
http://en.wikipedia.org/wiki/Quantile_function

However in Russian school books we have the other definition F(x) = P(X< x). This was the Russian standard during the time of the former Soviet Union, which is sometimes not compatible with DIN or ISO.
http://www.toehelp.ru/theory/ter_ver/3_2/

However in the Russian Wikipedia we find the new definition too
http://ru.wikipedia.org/wiki/Функция_распределения
http://www.exponenta.ru/educat/class/courses/tv/theme0/2.asp

In the Polish Wikipedia http://pl.wikipedia.org/wiki/Dystrybuanta e.g. you can read:

"Niech P będzie rozkładem prawdopodobieństwa na prostej. Funkcję F: R→R daną wzorem **F(t) = P((-∞,t])** nazywamy dystrybuantą rozkładu P. … Niekiedy w definicji dystrybuanty stosuje się przedział otwarty: **F(t) = P((-∞,t))** Dystrybuanta jest wówczas funkcją **lewostronnie ciągłą** (w przeciwieństwie do przypadku gdy w definicji stosuje się przedział prawostronnie domknięty i dystrybuanta jest funkcją **prawostronnie ciągłą**)."

Here the hint that both definitions exist – but what is the recommendation?

10. More references on ISO 3534-1 and ISO 31-11 around the world:

International standard, ISO 3534-1: Geneve, Switzerland : ISO, 2006.
Statistics - vocabulary and symbols. Part 1, Probability and general statistical terms = Statistique - vocabulaire et symboles. Partie 1, Probabilite et termes statistique generaux.

DIN ISO 3534-1, Berlin, Beuth-Verlag, 2009,
Titel (deutsch): Statistik - Begriffe und Formelzeichen - Teil 1: Wahrscheinlichkeit und allgemeine statistische Begriffe (ISO 3534-1:2006); Text Deutsch und Englisch

SANS 3534-1: Pretoria, South African national standard, 2007:
"This national standard is the identical implementation of **ISO 3534-1:2006** and is adopted with the permission of the International Organization for Standardization."
Cancels and replaces ed. 1 (SANS 3534-1/ISO 3534-1:1993), ISBN: 9780626189983

UNE-ISO 3534-1: Madrid, Asociación Española de Normalización y Certificación (AENOR) 2008, estadística : vocabulario y símbolos. Parte 1, Términos estadísticos generales y términos empleados en el cálculo de probabilidades

PN-ISO 3534-1: Warszawa : Polski Komitet Normalizacyjny 2002, Statystyka – Terminologia i symbole - Część 1: Ogólne terminy z zakresu rachunku prawdopodobieństwa i statystyki, ISBN: 9788323676690

SIST ISO 3534-1: Slovenski standard. Ljubljana: Slovenski inštitut za standardizacijo, 2008, Statistika - slovar in simboli. 1. del, Splošni statistični izrazi in izrazi v zvezi z verjetnostjo,

UNI ISO 3534-1 è la versione in lingua italiana, c.p.
http://it.wikipedia.org/wiki/UNI_ISO_3534-1

International standard, ISO 31-11, Geneve, Switzerland: ISO, 1992. Quantities and units = Grandeurs et unités. Part 11. = Partie 11, Mathematical signs and symbols for use in the physical sciences and technology = Signes et symboles mathématiques à employer dans les sciences physiques et dans la technique.

SIST ISO 31-11, Slovenski standard. Ljubljana: Urad Republike Slovenije za standardizacijo in meroslovje, cop. 2008.Veličine in enote. Del 11, Matematični znaki in simboli za uporabo v fizikalnih in tehniških vedah : (istoveten ISO 31-11:1992)

PN-ISO 31-11, Warszawa: Polski Komitet Normalizacyjny 2001, Wielkości fizyczne i jednostki miar - Znaki i symbole matematyczne do stosowania w naukach fizycznych i technice

New International standard, ISO 80000-2 (1st ed. 2009-12-01), Geneva, Switzerland: Quantities and units – Part 2: Mathematical signs and symbols to be used in the natural sciences and technology.
This edition cancels and replaces ISO 31-11 (1992), which has been technical revised.
Four clauses have been added, i.e. “Standard number sets and intervals”, “Elementary geometry”, “Combinatorics” and “Transforms”.