School-Mathematics all over the World – some Differences

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Abstract:
The lecture is devoted on some differences of definitions in school-mathematics. In other
countries we have sometimes other definitions compared with Germany. Sometimes we can
discover even in Germany some differences, because we have in Germany 16 smaller federal
countries and every country has its own ministry of education.
On the other hand every professor in any university is free and independent concerning the
research and education. He could use his own definition in special fields of mathematics. We
would have no problems, if we respect different definitions of different teachers and in
different software:

1. What is the set of natural numbers?
   \{0, 1, 2, \ldots\} or \{1, 2, 3, \ldots\} and what means the symbol \(\mathbb{N}\)?
2. What means \(3^{1/2}\)? Is it \(3 + 1/2 = 3.5\) or \(3 + 1/2 = 1.5\)?
3. Where the real function \(y = f(x) = x^{1/(3)}\) is defined? \(x\) must be non-negative or \(x \in \mathbb{R}\)?
4. What means the notation \(y = \tan^{-1}(x)\)? Is this the arctan-function or the cot-function?
5. What is \(y = \log(x)\)? We know \(y = \log_{\alpha}(x)\). What is the base \(\alpha\) in \(y = \log(x)\)?
   Without any \(\alpha\) we know \(y = \lg(x)\) or \(y = \ln(x)\) or \(y = \lg(x) = \lg(x)\) (older notation).
   Sometimes we find the notation \(y = \alpha \log(x)\). Why different notations?
6. What is the main argument \(\varphi\) of a complex number in the Gaussian plane?
   \[\varphi = \text{arg}(-2-2j) = -3\pi/4\] or \(\varphi = \text{arg}(-2-2j) = 5\pi/4\)?
7. What is the 3\(^{\text{rd}}\) main root of the complex number -2-2j? What is \((-2-2j)^{1/(3)}\)?
   What is \((-8)^{1/(3)}\)? Draw the real function \(y = f(x) = x^{1/(3)} = x^{0.3333333333}\)
8. What is the \(\alpha\)-quantile \(x_\alpha\) of a probability distribution of a random variable \(X\)?
   \(x_\alpha\) is a number with \(P(X < x_\alpha) \leq \alpha \leq P(X \leq x_\alpha)\) or \(P(X < x_\alpha) \leq 1-\alpha \leq P(X \leq x_\alpha)\)
9. Is the distribution function \(y = F(x)\) of a random variable \(X\) right continuous or left
   continuous? Is \(F(x) = P(X < x)\) or \(F(x) = P(X \leq x)\)? Consider the binomial distribution!

We have to solve in this context two basic problems:
A. If students change the school or university (this is the mobility of our young people) and
go to another country, than they remark or not remark that some differences exist in
mathematical definitions. In the examination sometimes we observe some mistakes
of our students and the reasons are some differences in education.

B. We observe sometimes differences between teaching and used software in calculators or
PC or used books of several authors.

In the lecture we will discuss about the stated problems and show some examples by the help
of CASIO ClassPad330 (operating system 3.06, published 2011).
The well known software package Mathematica (Version 8) by Wolfram could be the basic
for all teachers in the world to work with standard definitions which are used in Mathematica.
In Germany the “German Institute for Standardization” (Deutsches Institut für Normung,
DIN) offers stakeholders a platform for the development of standards as a service to industry,
the state and society as a whole. DIN has been based in Berlin since 1917. DIN's primary task
is to work closely with its stakeholders to develop consensus-based standards that meet
market requirements. By agreement with the German Federal Government, DIN is the
acknowledged national standards body that represents German interests in European and
international standards organizations. Ninety percent of the standards work now carried out
by DIN are international. Standards play a major deregulatory role. DIN’s goal is to develop standards that have validity worldwide.

**ISO (the International Organization for Standardization)** is the world's largest developer and publisher of **International Standards**. It has its headquarters in Geneva, Switzerland. ISO considers this trend of utmost importance and believes in the fundamental contribution that educational institutions can give on teaching what international standardization is and what can be achieved through it. Cp. chapter 10, **SANS** (South African national standard).

**References:**
http://www.din.de/cmd/?level=tpl-home&languageid=en  
http://www.iso.org/iso/home.htm  
http://edu.casio.com/products/classpad/  
http://www.wolfram.com/

1. **What is the set of natural numbers?**

There are two conventions for the set of natural numbers: it is either the set of positive integers \( \{1, 2, 3, \ldots\} \) according to the traditional definition; or the set of non-negative integers \( \{0, 1, 2, 3, \ldots\} \) according to a definition first appearing in the 19th century.

Have a look in **DIN5374** (logic and set theory, symbols and concepts) or **DIN1302** (general mathematical symbols and concepts) or **ISO31-11** (quantities and units – part 11: Mathematical signs and symbols for the use in physical sciences and technology, revised by ISO 80000-2:2009): \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \) and \( \mathbb{N}^* = \{1, 2, 3, \ldots\} \),  

E.g. in the online-book  
**MATHEMATICAL PREPARATION COURSE BEFORE STUDYING PHYSICS**  
[http://www.thphys.uni-heidelberg.de/~hefft/vk_download/vk1e.pdf](http://www.thphys.uni-heidelberg.de/~hefft/vk_download/vk1e.pdf) (March 10, 2011)  
you can read:  
“We begin with the set of natural numbers \( \{1, 2, 3, \ldots\} \), given the name \( \mathbb{N} \) by mathematicians and called “natural” because they have been used by mankind to count within living memory.” Here \( \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} \) (in Germany: ISBN 978-3-8274-1638-4)

2. **What means \( 3^{1/2} \)?**

![Image of a child solving a math problem](http://lamar.colostate.edu/~hillger/faq-images/faq-frac.jpg)

Using metric measurements, you don’t need to do arithmetic with mixed numbers.
What means \( 3^{1/2} \)? Often our students are not sure is it \( 3 + \frac{1}{2} = 3.5 \) or \( 3 \times \frac{1}{2} = 1.5 \)? For our students a problem of the mixed notation is that it can be misinterpreted as a product. A **mixed number** is the sum of a whole number and a proper fraction.

In the expression \( a^{b/c} \) is omitted the operator. Here it is a multiplication, because in terms with symbolic variables other arithmetic operators cannot be omitted, cp. DIN1302.

**The multiplication of quantity symbols** (or numbers in parentheses or values of quantities in parentheses) may be indicated in one of the following ways: \( ab, a \cdot b, a \times b \).

When the dot is used as the decimal marker as in the United States, the preferred sign for the multiplication of numbers or values of quantities is a cross (that is, multiplication sign) \( \times \), not a half-high (that is, centered) dot (\( . \)).

**Example:** Write \( 15 \times 72 \) but not \( 15 \cdot 72 \), cp. [http://physics.nist.gov/cuu/pdf/sp811.pdf](http://physics.nist.gov/cuu/pdf/sp811.pdf)

**In some countries such as France, the mixed number is unusual.**


You may, of course, *say* “three and a half” — 0.5 is often read aloud as “one half” — but you should always *write* it as a decimal fraction.

[http://larimar.comostate.edu/~hillger/faq.html](http://larimar.comostate.edu/~hillger/faq.html)

[http://larimar.comostate.edu/~hillger/decimal.htm](http://larimar.comostate.edu/~hillger/decimal.htm)

**Why Decimal?**

A room measures 15 ft. 3-3/4 in. by 21 ft. 7-1/2 in. (4.667 m by 6.591 m).

**Questions:**

- What is its floor area in *square yards*?
- What is its floor area in *square meters*?

**Answers:**

36.79 sq. yd., or 30.76 m²

**3. Where the real function \( y = f(x) = x^{1/3} \) is defined?**

The 3\(^{rd}\) root is a special case of exponentiation (real power with a fraction), cp. [http://en.wikipedia.org/wiki/Exponentiation](http://en.wikipedia.org/wiki/Exponentiation)

We know:

The exponentiation operation with integer exponents requires only elementary algebra.

By definition, raising a nonzero number to the \(-1\) power produces its reciprocal. Raising a positive real number \( x \) to a power that is not an integer, say \( 1/3 \), can be accomplished in two ways.

a) **Rational number exponents can be defined in terms of \( n^{th} \) roots, and arbitrary nonzero exponents can then be defined by continuity:**

\[ x^{\frac{1}{3}} = x^{0.333333333} = (x^{\frac{1}{3}})^{3} = (x^\frac{1}{1000000000})^{3} = (x^{\frac{1}{1000000000}})^{3} \]

(b) **The natural logarithm can be used to define real exponents using the exponential function:**

\[ x^{\frac{1}{3}} = \exp(\ln(x)/3) \]

Thus it is clear, \( x \) can't be negative (and not zero)

In DIN1302: \( y = x^{1/n} \) is the \( n^{th} \) root with a **positive** \( y \) such that \( y^n = x \). Here \( n \in \mathbb{N} \) and \( x \) is a nonnegative real number. In many German school books we find this definition of the \( n^{th} \) root in the domain of the real numbers.

e.g. Definition 1.6 in ISBN 978-3-427-21503-5(2007: Mathematik für Berufliche Gymnasien)
Another question is the real solution of the equation $x^3 = -8$.
The real solution is $-8^{1/3} = \frac{-8}{-8} = -2$ but not $(-8)^{1/3}$ or $(-8)^0.333333333$.
The last numbers are not real but complex (cp. principal value, complex main root in C).

In US-school books a negative base is allowed:
For any real numbers $a$ and $b$, and any positive integer $n$, if $a^n = b$, then $a$ is the $n^{th}$ root of $b$.
Here $(-8)^{1/3} = -2$, where from the point of view of C the value -2 is not the principal root.

http://www.farmersville.k12.ca.us/aztecs/Department/Math/spraudling/Algebra%20I/Alg%20II/unit%2017/Review%20&%20tests/Ch%202/Review%20answers.pdf


4. What means the notation $y = \tan^{-1}(x)$?
Is this the arctan-function or the cot-function?
For a given function $y = f(x)$ we have the notation $y = f^{-1}(x)$ for the inverse function.
However the notation $(f(x))^{-1}$ means $1/f(x)$. Here we have the exponentiation with -1.

In textbooks we often read e.g. $\sin^2(x) + \cos^2(x) = 1$ (trigonometric Pythagoras)

If you work with a calculator: the input $\sin^2(x) + \cos^2(x) = 1$ is not defined. You have to write $(\sin(x))^2 + (\cos(x))^2 = 1$ or you have to define the symbol $\sin^2$ and $\cos^2$ (a name with four characters!). On the other hand the symbol $\tan^{-1}$ is clear for the calculator.

Only in the case if the symbolic exponent equals -1 our students have problems e.g. with $y = \tan^{-1}(x)$. The calculators use the notation $\tan^{-1}(x)$, but in written form the students should use the notation $\arctan(x)$ (DIN1302) and not $\tan^{-1}(x)$ to avoid problems and misunderstandings.

5. What is $y = \log(x)$?
We know $y = \log_a(x)$. What is the base $a$ in $y = \log(x)$?
Without any $a$ we know $y = \log(x)$ or $y = \ln(x)$ or $y = \log_b(x) = \text{Id}(x)$ (older notation). Sometimes we find the notation $y = a\log(x)$. Why different notations? We should use modern notations given by DIN and ISO. We should avoid the old notations $\text{Id}(x)$ and $a\log(x)$. In calculators $\log(x)$ means $\lg(x)$. Why not a $\lg$-key instead of $\log$-key? In ISBN 978-3-8274-1638-4 (and English and Spanish translation 2011) again the old notation, e.g. $\text{Id}(x) = a\log(x)$ – why?
6. What is the main argument $\varphi$ of a complex number in the Gaussian plane?

$$\varphi = \arg(-2-2j) = -3\pi/4 \text{ or } \varphi = \arg(-2-2j) = 5\pi/4?$$

In every calculators and PC-software and DIN1302 the main argument is in the interval $[-\pi, \pi] = (-\pi, \pi]$, but in many school books we can read: the main argument is in the interval $[0, 2\pi] = [0, 2\pi)$.

**Mathematica:**

```math
In[1]:= Sqrt[-1]
Out[1]= i
In[2]:= Arg[-2-2i]
Out[2]= -3\pi/4
```

**ClassPad:**

$$\sqrt{-1} = j \quad \arg(-2-2j) = -3\pi/4$$

We have two notations for the complex unit: $i$ in mathematics and $j$ in engineering.

In the engl. translation (2011) of the book ISBN 978-3-8274-1638-4 you can read:

“for the (only modulo 2\pi determined) argument of a complex number:

$$0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi$$

Here $z = x + yj$, i.e. $(x, y)$ are the Cartesian coordinates of $z$. Why here not include $\varphi = 0$?

We know the real function $f(t) = \arctan(t)$ has its values in the interval $]-\pi/2, \pi/2[ = (-\pi/2, \pi/2)$, thus the definition $0 < \varphi = \arg z := \arctan(y/x) \leq 2\pi$ is not correct!

In the German online script http://www.thphys.uni-heidelberg.de/~hefft/vk_download/vk1.pdf (March 11th, 2011) the author has added a remark:

„Einschub: Alternative Phasenkonvention:“ Natürlich kann man auch ein um den Ursprung symmetrisches Intervall der Länge 2\pi für die Argumente der komplexen Zahlen wählen:

$$-\pi < \varphi = \arg z := \arctan(y/x) \leq \pi,$$ was allerdings später die Herleitung der Wurzelfunktionen etwas komplizierter macht. Nach Prof. L. Paditz von der HTW Dresden empfiehlt die DIN diese Konvention."

Here the author is again not correct: He has to write $-\pi < \varphi = \arg z \leq \pi$ whithout the arctan-function! Furthermore he remarks that with the DIN convention for the argument $\varphi$ the definition of the root-function will be more difficult – this is not true! The definition of the root based on the main argument $\varphi = \arg z$ is very easy:

The main root is $z_0 = z^\Delta (1/n) = |z|^\Delta (1/n) \cdot \exp(\arg(z)/n)$

and the other roots are $z_k = z_0 \cdot \exp(k \cdot 2\pi i/n)$, $k = 1, 2, \ldots, n-1$.

This definition of the main root and the other roots is implemented in all calculators and PC-software.

**ClassPad:**

$\arg(x+yi)|x<0$ yields the correct main argument $\tan^{-1}(y/x)+\text{signum}(y)\cdot\pi$

Spanish version (March 11th, 2011) of the considered online book „MATHEMATICAL PREPARATION COURSE BEFORE STUDYING PHYSICS“:

CURSO DE MATEMÁTICA PREPARATORIO - para el estudio de la Física traducido por Prof. Dr. LAUTARO VERA, Universidad de Santiago de Chile

http://www.thphys.uni-heidelberg.de/~hefft/vk1/k0/000s.htm

“Argumento el número complejo: $0 < \varphi = \arg z := \arctan(y/x) < 2\pi$ (sólo determinado módulo $2\pi$)”

http://www.thphys.uni-heidelberg.de/~hefft/vk1/k8/814s.htm

Thus an obvious error goes around the world!
7. What is the 3\textsuperscript{rd} main root of the complex number -2-2\textit{j}?

What is (-2-2\textit{j})\textsuperscript{4/3}?

What is (-8)\textsuperscript{4/3}? Draw the real function \( y = f(x) = x^{4/3} = x^{0.33333333} \) !

The main root is \( z_0 = z^{4/3} = |z|^{4/3} \ast \exp(\text{arg}(z)j/3) = 1 - j \) if \( z = -2-2\textit{j} \)
Here we use the main argument \( \text{arg}(z) = -3\pi/4 \), i.e. \( \text{arg}(z)j/3 = -\pi/4 \).

If we follow the other point of view with \( \text{arg}(z) = 5\pi/4 \), i.e. \( \text{arg}(z)j/3 = 5\pi/12 \), we would get the main root \( (-2-2\textit{j})^{4/3} = \sqrt[3]{3}/2 - 1/2 + (\sqrt[3]{3}/2 + 1/2) \textit{j} \) and not -1-\textit{j}.

The same with (-8)\textsuperscript{4/3}. The main root is \( 2\exp(\pi j/3) = 1 + \sqrt[3]{3} \textit{j} \) and not -2.

\textbf{ClassPad: in complex mode:} (-8)\textsuperscript{4/3} = 1 + \sqrt[3]{3} \textit{j}; \textbf{in real mode:} (-8)\textsuperscript{4/3} = -2.

In the real mode the ClassPad follows the US-definition of a root, thus we get here different graphics of \( y = f(x) = x^{4/3} \) in the x-y-plane, in dependence on real mode or complex mode:

However in real mode or complex mode we get the same graphics for \( y = f(x) = x^{0.33333333} \!\).

8. What is the \( \alpha \)-quantile \( x_\alpha \) of a probability distribution of a random variable \( X \)?

\( x_\alpha \) is a number with \( P(X < x_\alpha) < \alpha \leq P(X < x_\alpha) \) or \( P(X < x_\alpha) \leq 1-\alpha \leq P(X < x_\alpha) \) ?

Follow \textbf{Mathematica} “Quantile[dist, \( \alpha \)] is equivalent to InverseCDF[dist, \( \alpha \)]” we get the definition: \( P(X < x_\alpha) < \alpha \leq P(X < x_\alpha) \), i.e. according to DIN ISO 3534-1 (2009) (Statistics – Vocabulary and symbols – Part 1: General statistical terms and terms used in probability) based on ISO 3534-1 (2006) (http://it.wikipedia.org/wiki/ISO_3534).

For a continuous distribution of \( X \) the inverse CDF at \( \alpha \) is the value \( x_\alpha \) such that \( CDF[dist, x_\alpha]= \alpha \). For a discrete distribution of \( X \) the inverse CDF at \( \alpha \) is the smallest integer \( x_\alpha \) such that \( CDF[dist, x_\alpha] \geq \alpha \).

\textbf{cp.}

http://reference.wolfram.com/mathematica/ref/InverseCDF.html
http://reference.wolfram.com/mathematica/ref/Probability.html
However ClassPad only follows this definition for discrete distributions but for continuous distributions ClassPad works with the other definition \( P(X < x_\alpha) = 1 - \alpha \leq P(X \leq x_\alpha) \).

The last picture shows a good service during the computing the quantile of a discrete distribution. If the assumed probability (say \( \alpha = 0.05 \)) is no value of the distribution function, than the computed quantile is the smallest integer \( x_\alpha \) such that \( \alpha \leq P(X \leq x_\alpha) = F(x_\alpha), \) i.e. for \( x_\alpha - 1 \) we have already \( F(x_\alpha - 1) = P(X \leq x_\alpha - 1) = P(X < x_\alpha) < \alpha \). The “WARNING!” is connected with the last digit after the decimal point of the assumed probability \( \text{prob} = \alpha \) and will not appear, if a change in the last digit by one step will not change the computed quantile.

9. Is the distribution function \( y = F(x) \) of a random variable \( X \) right continuous or left continuous?

Is \( F(x) = P(X < x) \) or \( F(x) = P(X \leq x) \)? Consider the binomial distribution!

For a continuous distribution both definitions of \( F(x) \) are equal, but for discrete distributions we get here a left continuous function with \( F(x) = P(X < x) \) and a right continuous function with \( F(x) = P(X \leq x) \). The last definition is in context with DIN ISO 3534-1 (2009) based on ISO 3534-1 (2006) and all software, e.g. Mathematica or ClassPad.

http://en.wikipedia.org/wiki/Cumulative_distribution_function
http://en.wikipedia.org/wiki/Quantile_function

However in Russian school books we have the other definition \( F(x) = P(X < x) \). This was the Russian standard during the time of the former Soviet Union, which is sometimes not compatible with DIN or ISO.

http://www.toehelp.ru/theory/ter_ver/3_2/

However in the Russian Wikipedia we find the new definition too
http://ru.wikipedia.org/wiki/Функция_распределения
http://www.exponenta.ru/educat/class/courses/tv/theme0/2.asp

In the Polish Wikipedia http://pl.wikipedia.org/wiki/Dystrybuanta e.g. you can read:

“Niez P będzie rozkładem prawdopodobieństwa na prostej. Funkcję \( F: R \rightarrow R \) daną wzorem \( F(t) = P((-\infty,t]) \) nazywamy dystrybuantą rozkładu \( P \). … Nickiedy w definicji dystrybuanty stosuje się przedział otwarty: \( F(t) = P((-\infty,t)) \) Dystrybuanta jest wówczas funkcją lewostronne ciagiłą (w przeciwieństwie do przypadku gdy w definicji stosuje się przedział prawostronne domknięty i dystrybuanta jest funkcją prawostronne ciagiłą).”

Here the hint that both definitions exist – but what is the recommendation?
10. More references on ISO 3534-1 and ISO 31-11 around the world:

Statistics - vocabulary and symbols. Part 1, Probability and general statistical terms = Statistique - vocabulaire et symboles. Partie 1, Probabilite et termes statistique generaux.

**DIN ISO 3534-1**, Berlin, Beuth-Verlag, 2009,
Titel (deutsch): Statistik - Begriffe und Formelzeichen - Teil 1: Wahrscheinlichkeit und allgemeine statistische Begriffe (ISO 3534-1:2006); Text Deutsch und Englisch

**SANS 3534-1**: Pretoria, South African national standard, 2007:
"This national standard is the identical implementation of ISO 3534-1:2006 and is adopted with the permission of the International Organization for Standardization."

**UNE-ISO 3534-1**: Madrid, Asociación Española de Normalización y Certificación (AENOR) 2008, estadística : vocabulario y símbolos. Parte 1, Términos estadísticos generales y términos empleados en el cálculo de probabilidades


**SIST ISO 3534-1**: Slovenski standard. Ljubljana: Slovenski inštitut za standardizacijo, 2008, Statistika - slovar in simboli. I. del, Splošni statistični izrazi in izrazi v zvezi z verjetnostjo,

**UNI ISO 3534-1** è la versione in lingua italiana, c.p.
http://it.wikipedia.org/wiki/UNI_ISO_3534-1


**PN-ISO 31-11**, Warszawa: Polski Komitet Normalizacyjny 2001, Wielkości fizyczne i jednostki miar - Znaki i symbole matematyczne do stosowania w naukach fizycznych i technice

**New International standard, ISO 80000-2** (1st ed. 2009-12-01), Geneva, Switzerland:
Quantities and units – Part 2: Mathematical signs and symbols to be used in the natural sciences and technology.
This edition cancels and replaces ISO 31-11 (1992), which has been technical revised.
Four clauses have been added, i.e. “Standard number sets and intervals”, “Elementary geometry”, “Combinatorics” and “Transforms”. 